MODULAR IDENTITIES FOR THE ROGERS–RAMANUJAN FUNCTIONS AND ANALOGUES

BY

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DISSERTATION

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Abstract

In his notebooks, Ramanujan recorded 40 beautiful modular relations for the Rogers–Ramanujan functions. Of these 40 identities, precisely one involves powers of the Rogers–Ramanujan functions. Ramanujan added the enigmatic note [74], [77, p. 231] that "Each of these formulae is the simplest of a large class." This suggests that there are further modular identities involving powers of the Rogers–Ramanujan functions. Although numerous authors have studied identities for the Rogers–Ramanujan functions and various analogues, no systematic study of identities involving powers of the Rogers–Ramanujan functions has been undertaken. In this thesis, we continue the study of modular identities for the Rogers–Ramanujan functions, with particular emphasis on relations involving powers of the Rogers–Ramanujan functions. Our methods are classical, using tools and techniques that Ramanujan could have employed. These tools include, for example, manipulation of infinite series and the theory of modular equations. It is hoped that these methods will give new insights into these equations, and perhaps lead to understanding or discovering further families of identities of mathematical interest.

Identities involving squares, cubes, fourth, and fifth powers of the Rogers–Ramanujan functions are enunciated and proved; many of these relations are new. Rich applications are made to the study of modular relations for the Rogers–Ramanujan continued fraction. To demonstrate the generality of our methods, analogous results are obtained in various cases for the Göllnitz–Gordon functions and the Ramanujan–Göllnitz–Gordon continued fraction.

Further identities for the Rogers–Ramanujan functions, of the types found in Ramanujan's list of 40 relations for the Rogers–Ramanujan functions, are also studied. Analogous identities are obtained for the Göllnitz–Gordon functions, as well as for dodecic and sextodecic analogues of the Rogers–Ramanujan functions.

To my Family –

To my Mother, Father, and Brother

and

To my loving wife, Mei-Chuang Kuo

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Chapter 1

Introduction

We begin by introducing Ramanujan's definition for a general theta function, standard notation, and basic results. Here and in the sequel we assume that $|q| < 1$, and employ the customary q–product notation. Thus, set

$$(a;q)_0 := 1,$$
$$(a;q)_n := \prod_{k=0}^{n-1}(1 - aq^k), \quad n = 1, 2, ...,$$

and

$$(a;q)_\infty := \prod_{k=0}^{\infty}(1 - aq^k).$$

Ramanujan's general theta function is defined by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \qquad |ab| < 1. \tag{1.0.1}$$

Basic properties satisfied by $f(a,b)$ include [14, p. 34, Entry 18]

$$f(a,b) = f(b,a), \tag{1.0.2}$$
$$f(1,a) = 2f(a,a^3), \tag{1.0.3}$$
$$f(-1,a) = 0, \tag{1.0.4}$$

and, if n is an integer,

$$f(a,b) = a^{n(n+1)/2} b^{n(n-1)/2} f(a(ab)^n, b(ab)^{-n}). \tag{1.0.5}$$

Property (1.0.2) is used frequently and without mention in the sequel. The function $f(a,b)$ satisfies the well-known Jacobi Triple Product Identity [14, Entry 19, p. 35]

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty. \tag{1.0.6}$$

The three most important special cases of (1.0.6) are

$$\phi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2 (q^2;q^2)_\infty, \tag{1.0.7}$$

$$\psi(q) := f(q, q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{1.0.8}$$

and

$$\begin{aligned} f(-q) : = f(-q, -q^2) &= \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} \\ &= (q;q)_\infty =: q^{-1/24}\eta(z), \end{aligned} \tag{1.0.9}$$

where $q = e^{2\pi i z}$, $\operatorname{Im} z > 0$, and $\eta(z)$ is the classical Dedekind-eta function. The penultimate equality in (1.0.9) is Euler's pentagonal number theorem.

After Ramanujan, define

$$\chi(q) := (-q;q^2)_\infty. \tag{1.0.10}$$

We often find it convenient to employ the notation

$$f_n := f(-q^n). \tag{1.0.11}$$

We record here a useful lemma, which follows from the Jacobi Triple Product Identity (1.0.6) and the definitions (1.0.7)–(1.0.11) above.

Lemma 1.0.1.

$$\phi(-q) = \frac{f_1^2}{f_2}, \qquad \psi(q) = \frac{f_2^2}{f_1}, \qquad \chi(-q) = \frac{f_1}{f_2}, \tag{1.0.12}$$

and

$$\phi(q) = \frac{f_2^5}{f_1^2 f_4^2}, \qquad \psi(-q) = \frac{f_1 f_4}{f_2}, \qquad \chi(q) = \frac{f_2^2}{f_1 f_4}, \qquad f(q) = \frac{f_2^3}{f_1 f_4}. \tag{1.0.13}$$

In the sequel, we shall require a useful expansion formula for $f(a,b)$. For each nonnegative integer n, let

$$U_n := a^{n(n+1)/2} b^{n(n-1)/2} \qquad \text{and} \qquad V_n := a^{n(n-1)/2} b^{n(n+1)/2}. \tag{1.0.14}$$

Then [14, p. 48, Entry 31]

$$f(a,b) = \sum_{r=0}^{n-1} U_r f\left(\frac{U_{n+r}}{U_r}, \frac{V_{n-r}}{U_r}\right). \tag{1.0.15}$$

We also have the following elementary results [14, p. 45, Entry 29]. If $ab = cd$, then

$$f(a,b)f(c,d) + f(-a,-b)f(-c,-d) = 2f(ac,bd)f(ad,bc), \tag{1.0.16}$$

$$f(a,b)f(c,d) - f(-a,-b)f(-c,-d) = 2af\left(\frac{b}{c}, \frac{c}{b}abcd\right) f\left(\frac{b}{d}, \frac{d}{b}abcd\right). \tag{1.0.17}$$

Adding (1.0.16) and (1.0.17), we deduce

Lemma 1.0.2. *If $ab = cd$, then*

$$f(a,b)f(c,d) = f(ac,bd)f(ad,bc) + af\left(\frac{b}{c}, \frac{c}{b}abcd\right) f\left(\frac{b}{d}, \frac{d}{b}abcd\right). \qquad (1.0.18)$$

D. Hickerson utilized an equivalent form of Lemma 1.0.2 [52, Theorem 1.1] in his important work proving the mock theta conjectures.

The function $f(a,b)$ satisfies the fundamental Quintuple Product Identity [14, p. 80], a version of which can be found on page 207 of Ramanujan's Lost Notebook. For $B \neq 0$,

$$f(B^3q, q^5/B^3) - B^2 f(q/B^3, B^3q^5) = f(-q^2)\frac{f(-B^2, -q^2/B^2)}{f(Bq, q/B)}. \qquad (1.0.19)$$

For a history of this identity, see the work of S. Cooper [38]. Recently, S. Kim [62] has given a bijective proof of (1.0.19).

H. Schröter [14, pp. 65–72] developed useful representations for a product of two theta functions as a sum of m products of pairs of theta functions. An elegant generalization of Schröter's work has been discovered by R. Blecksmith, J. Brillhart, and I. Gerst [26, Theorem 2]. We translate their formula into Ramanujan's notation. Define, for $\epsilon \in \{0,1\}$ and $|ab| < 1$,

$$f_\epsilon(a,b) = \sum_{n=-\infty}^{\infty} (-1)^{\epsilon n}(ab)^{n^2/2}(a/b)^{n/2}.$$

Theorem 1.0.3. [26, Theorem 2] *Let a, b, c, and d denote positive numbers with $|ab|$, $|cd| < 1$. Suppose that there exist positive integers α, β, and m such that*

$$(ab)^\beta = (cd)^{\alpha(m-\alpha\beta)}.$$

Let $\epsilon_1, \epsilon_2 \in \{0,1\}$, and define $\delta_1, \delta_2 \in \{0,1\}$ by

$$\delta_1 \equiv \epsilon_1 - \alpha\epsilon_2 \pmod 2 \qquad \text{and} \qquad \delta_2 \equiv \beta\epsilon_1 + p\epsilon_2 \pmod 2$$

respectively, where $p = m - \alpha\beta$. Then, if R denotes any complete residue system modulo m,

$$\begin{aligned} &f_{\epsilon_1}(a,b)f_{\epsilon_2}(c,d) = \\ &\quad \sum_{r\in R}(-1)^{\epsilon_2 r}c^{r(r+1)/2}d^{r(r-1)/2}f_{\delta_1}\left(\frac{a(cd)^{\alpha(\alpha+1-2r)/2}}{c^\alpha}, \frac{b(cd)^{\alpha(\alpha+1+2r)/2}}{d^\alpha}\right) \\ &\qquad\qquad \times f_{\delta_2}\left(\frac{(b/a)^{\beta/2}(cd)^{p(m+1-2r)/2}}{c^p}, \frac{(a/b)^{\beta/2}(cd)^{p(m+1+2r)/2}}{d^p}\right). \end{aligned}$$

We note that Z. Cao [31, 32] has obtained a fascinating generalization of Schröter's formulas. His generalization contains as a special case Theorem 1.0.3.

To prove some of our results, we use the theory of modular equations. There are many definitions of a modular equation in the literature. For example, see the books by R.A. Rankin [79, p. 76] and B. Schoeneberg [85, pp. 142–144]. We give the definition of modular equation as understood by Ramanujan. To that end, we first define the complete elliptic integral of the first kind, $K(k)$, by [100, pp. 499-500]

$$K := K(k) := \int_0^{\pi/2} \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}}$$
$$= \frac{\pi}{2}\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^2}{(n!)^2} k^{2n} = \frac{\pi}{2}\, {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; k^2\right), \quad (0 < k < 1),$$

where ${}_2F_1$ denotes the ordinary or Gaussian hypergeometric function,

$$(\alpha)_0 := 1,$$

and

$$(\alpha)_n := \alpha(\alpha+1)\cdots(\alpha+n-1)$$

for each nonnegative integer n. The number k is called the modulus of K, and $k' := \sqrt{1-k^2}$ is called the complementary modulus.

Let K, K', L, and L' denote complete elliptic integrals of the first kind associated with the moduli k, k', l, and l', respectively, where $0 < k, l < 1$. Suppose that

$$\frac{L'}{L} = n\frac{K'}{K} \tag{1.0.20}$$

holds for some positive integer n. A relation between k and l induced by (1.0.20) is called a modular equation of degree n. If we set

$$q = \exp\left(-\pi\frac{K'}{K}\right) \qquad \text{and} \qquad q' = \exp\left(-\pi\frac{L'}{L}\right),$$

we see that (1.0.20) is equivalent to the relation $q^n = q'$. Thus, a modular equation can be viewed as an identity involving theta functions at the arguments q and q^n.

Following Ramanujan, set

$$\alpha = k^2 \qquad \text{and} \qquad \beta = l^2.$$

We often say that β has degree n over α. If $q = \exp(-\pi K'/K)$, one of the most fundamental relations in the theory of elliptic functions is given by the set of formulas [14, pp. 101-102]

$$\phi^2(q) = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; k^2\right) = \frac{2}{\pi}\int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = \frac{2}{\pi}K(k)$$

and

$$\alpha = k^2 = 1 - \frac{\phi^4(-q)}{\phi^4(q)},$$

where $0 < k < 1$. Thus, an evaluation of any one of the functions ϕ, ${}_2F_1$, or K, yields an evaluation of the other two functions.

The multiplier m for a modular equation of degree n is defined by

$$m := \frac{K}{L} = \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \alpha\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \beta\right)} = \frac{z_1}{z_n} = \frac{\phi^2(q)}{\phi^2(q^n)}, \tag{1.0.21}$$

where

$$z_r := \frac{2}{\pi}K(k) = \phi^2(q^r).$$

Ramanujan derived an extensive catalogue of formulas [14, pp. 122-124] giving "evaluations" of $f(\pm q)$, $\phi(\pm q)$, $\psi(\pm q)$, and $\chi(\pm q)$ at various powers of the arguments in terms of

$$z := z_1 = \frac{2}{\pi}K(k), \qquad \alpha, \qquad \text{and} \qquad q.$$

If q is replaced by q^n, then the evaluations are given in terms of

$$z_n := \frac{2}{\pi}K(l), \qquad \beta, \qquad \text{and} \qquad q^n,$$

where β has degree n over α. Ramanujan also established many modular equations of "mixed" degree, in which four distinct moduli appear. We require in the sequel the following evaluations:

Lemma 1.0.4. [14, pp. 123–124, Entries 11(i),(iv); 12(ii),(iii),(iv),(vi)] *The following identities hold:*

$$\psi(q) = \sqrt{\frac{1}{2}z_1}\left(\alpha\frac{1}{q}\right)^{1/8}, \qquad \psi(q^4) = \frac{1}{2}\sqrt{\frac{1}{2}z_1}\left(\left(1-\sqrt{1-\alpha}\right)\frac{1}{q}\right)^{1/2},$$

$$f(-q) = \sqrt{z_1}2^{-1/6}\left(1-\alpha\right)^{1/6}\left(\alpha\frac{1}{q}\right)^{1/24},$$

$$f(-q^2) = \sqrt{z_1}2^{-1/3}\left(\alpha\left(1-\alpha\right)\frac{1}{q}\right)^{1/12},$$

$$f(-q^4) = \sqrt{z_1}4^{-1/3}\left(1-\alpha\right)^{1/24}\left(\alpha\frac{1}{q}\right)^{1/6},$$

and

$$\chi(-q) = 2^{1/6}\left(1-\alpha\right)^{1/12}\left(\alpha\frac{1}{q}\right)^{-1/24}.$$

Next, we define two of the central objects of study considered in our work. The Rogers–Ramanujan functions are defined by

$$G(q) := \sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q;q^5)_\infty(q^4;q^5)_\infty} \tag{1.0.22}$$

and

$$H(q) := \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q;q)_n} = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty}. \tag{1.0.23}$$

The product-series equalities in (1.0.22) and (1.0.23) are the celebrated Rogers–Ramanujan identities, first proved by L.J. Rogers [81] in 1894 (see also [72], [77, pp. 214–215]). At the end of his brief communication [74], [77, p. 213] announcing his proofs of the Rogers–Ramanujan identities, Ramanujan remarks, "I have now found an algebraic relation between $G(q)$ and $H(q)$, viz:

$$H(q)\left\{G(q)\right\}^{11} - q^2 G(q)\left\{H(q)\right\}^{11} = 1 + 11q\left\{G(q)H(q)\right\}^6. \tag{1.0.24}$$

Another noteworthy formula is

$$H(q)G(q^{11}) - q^2 G(q)H(q^{11}) = 1. \tag{1.0.25}$$

Each of these formulae is the simplest of a large class."

In all, Ramanujan [78, pp. 236-237] recorded 40 beautiful modular relations for the Rogers–Ramanujan functions. He did not, however, give any indication of how one might prove them. Motivated by Ramanujan's list, a number of mathematicians have attempted to understand and prove these relations. In his paper [82] establishing ten of the identities, Rogers remarks, "these [identities] were communicated privately to me in February 1919...". Rogers did not indicate if further identities were included in Ramanujan's communication to him.

In 1933, G.N. Watson [99] proved eight of the identities, but with two of them from the group that Rogers had proved. Watson confided, "Among the formulae contained in the manuscripts left by Ramanujan is a set of about forty which involve functions of the types $G(q)$ and $H(q)$; the beauty of these formulae seems to me to be comparable with that of the Rogers–Ramanujan identities. So far as I know, nobody else has discovered any formulae which approach them even remotely; if my belief is well-founded, the undivided credit for the discovery of these formulae is due to Ramanujan."

Ramanujan's forty identities for $G(q)$ and $H(q)$ were first brought in their entirety before the mathematical public by B.J. Birch [25], who in 1975 found Watson's handwritten copy of Ramanujan's list of forty identities in the Oxford University Library. Ramanujan's original manuscript was in Watson's possession for many years and now has evidently been lost. Watson's handwritten list was later published along with Ramanujan's lost notebook [78, pp. 236–237] in 1988.

As mentioned previously, Rogers [82] established ten of the 40 identities. After Watson's paper, a total of sixteen had thus been proved. D. Bressoud [27], in his Ph.D. thesis, proved fifteen from the list of forty by combining and extending the methods of Rogers and Watson. His published paper [28] contains proofs of some, but not all, of the general identities from [27] that he developed in order to prove Ramanujan's identities. All of the proofs of Rogers, Watson, and Bressoud employ classical techniques, although it is likely in most cases that the proofs found differ from Ramanujan's approaches.

After the work of Rogers, Watson, and Bressoud, nine identities remained to be proved. A.J.F. Biagioli [24] used the theory of modular forms to prove eight of them; it is clear that modular forms can also be used to establish the last identity. Recently, B. Berndt, G. Choi, Y.–S. Choi, H. Hahn, B.P. Yeap, A.J. Yee, H. Yesilyurt, and J. Yi [19] authored a beautiful memoir on the 40 identities in which they offered many new proofs. In [101], Yesilyurt provided for the first time classical proofs for two more of the 40 identities. Currently, three identities from Ramanujan's list of 40 identities still have not been proved by methods Ramanujan might have employed. It would be of great interest to understand and prove these by such methods, as this may provide insights into techniques that could be used to give easier proofs of important identities, or allow us to discover new families of identities of interest to the mathematical community.

In addition to Ramanujan's list of 40 identities, recent further work has been undertaken to find new relations for the Rogers–Ramanujan functions. In his Ph.D. thesis [80], S. Robins utilized the theory of modular forms to discover 4 new relations for the Rogers–Ramanujan functions, as well as 9 relations for various analogues of the Rogers–Ramanujan functions. Two of Robins' identities for the Rogers–Ramanujan functions were independently rediscovered by B. Gordon and R.J. McIntosh [46] in the context of studying transformation formulas for mock theta functions; Gordon and McIntosh provided elementary proofs.

Motivated by connections to Thompson series, M. Koike [63] used a computer to discover empirically several new relations for $G(q)$ and $H(q)$; his searches also yielded many identities already on Ramanujan's list. Some of Koike's identities also expressed relations for the Rogers–Ramanujan functions directly in terms of Thompson series. In a similar spirit, M. Somos [93] has used PARI-GP code that he has written to find new relations for the Rogers–Ramanujan functions.

K. Bringmann and H. Swisher [29, 30] have put Koike's conjectures on a rigorous foundation by using the theory of modular forms to prove them. Recently, Berndt and Yesilyurt [23] employed an idea of Watson in order to discover and prove a large number of new relations for $G(q)$ and $H(q)$. A new, isolated identity was found by S.–S. Huang [58].

The techniques of Robins, Bringmann, and Swisher involve the theory of modular forms, and the methods of Koike and Somos are computational and yield conjectures only. In this thesis, we use classical and elementary methods in the spirit of Ramanujan in order to give new proofs of many of the results of Robins, Koike, Somos, Bringmann, and Swisher. Several new identities are also presented. A special emphasis is placed on identities involving powers of the Rogers–Ramanujan functions, which we discuss in more detail next.

Among Ramanujuan's list of 40 identities, only one, (1.0.24), involves powers of the Rogers–Ramanujan functions. As related above, Ramanujan [74] claims that "Each of these formulae is the simplest of a large class." This suggests that there should be further identities involving powers of the Rogers–Ramanujan functions. Despite this, no systematic study of identities involving powers of the Rogers–Ramanujan functions has previously been undertaken, and only three further identities involving powers of the Rogers–Ramanujan functions had, prior to our work, been discovered. A major goal of this thesis is a systematic study of identities involving powers of the Rogers–Ramanujan functions. Identities for squares, cubes, fourth, and fifth powers of the Rogers–Ramanujan functions are considered.

Intimately connected to the Rogers–Ramanujan functions is the famous Rogers–Ramanujan continued fraction, defined by

$$R(q) := \frac{q^{1/5}}{1} \, \genfrac{}{}{0pt}{}{}{+} \, \frac{q}{1} \, \genfrac{}{}{0pt}{}{}{+} \, \frac{q^2}{1} \, \genfrac{}{}{0pt}{}{}{+} \, \frac{q^3}{1} \, \genfrac{}{}{0pt}{}{}{+} \, \genfrac{}{}{0pt}{}{}{\cdots}, \qquad |q| < 1. \tag{1.0.26}$$

The continued fraction $R(q)$ first appeared in a paper by Rogers [81] in 1894. Ramanujan later rediscovered and extensively developed the theory of $R(q)$ [76, 78]. This continued fraction has since been extensively studied and applied. See, for example, [7] for references.

In his notebooks, Ramanujan recorded several modular identities connecting the Rogers–Ramanujan continued fraction at various powers of the argument q. In this thesis, we provide new proofs of these identities. Our primary technique is to connect the relations for the Rogers–Ramanujan continued fraction with identities for the Rogers–Ramanujan functions. Many of the identities for the Rogers–Ramanujan functions that we employ involve powers of the Rogers–Ramanujan functions. Our approach is perhaps the most interesting of all the approaches that have been employed to prove these identities, in that we often provide previously unknown factorizations of the identities, or, when quotients are involved, hitherto unknown identities for the expressions appearing in the numerators and denominators.

In connection with $R(q)$, Ramanujan adroitly defined the following parameters.

Definition 1.0.5. *Define*

$$k := k(q) := R(q)R^2(q^2), \tag{1.0.27}$$

$$\mu := \mu(q) := R(q)R(q^4), \tag{1.0.28}$$

and

$$\nu := \nu(q) := \frac{R^2(q^{1/2})R(q)}{R(q^2)}. \tag{1.0.29}$$

In his notebooks, Ramanujan recorded a number of identities involving k, μ, and ν. S.–Y. Kang [59, 60] was the first to systematically investigate many of these relations. Using identities for the Rogers–Ramanujan functions, we give new proofs for several of the identities for k, μ, and ν. Applications of these parameters are provided to theta function identities and to the Rogers–Ramanujan continued fraction.

Although Ramanujan recorded a large number of results for k, he recorded far fewer results for the parameters μ and ν. Inspired by Ramanujan's identities for k, Kang [59] derived new analogues for μ and ν for some of Ramanujan's relations for k. We show by our approach that we can naturally obtain a full suite of results for μ and ν that extends the work of Kang and parallels the theory for the parameter k.

L.J. Slater [89] discovered a very large class of analogues of the Rogers–Ramanujan functions. Some of these analogues have been studied in detail. For example, Huang [58] has derived a list of identities for the Göllnitz–Gordon functions that is comparable to Ramanujan's list of forty identities for the Rogers–Ramanujan functions. N.D. Baruah, J. Bora, and N. Saika [13] have given new proofs of most of the relations that Huang found; their methods have yielded new identities as well. In 2003, H. Hahn [51] derived a large number of identities for septic analogues of the Rogers–Ramanujan functions. Nonic and dodecic analogues of the Rogers–Ramanujan functions have been studied by Baruah and Bora [11, 12]. In his thesis [80], Robins also derived identities for the Göllnitz–Gordon functions and the dodecic analogues.

In 1987, Andrews [6] showed how one could extract combinatorial information out of relations for the Rogers–Ramanujan functions utilizing the notion of colored partitions. Huang [58], Hahn [51], and Baruah and Bora [11, 12] have applied Andrews' ideas in their studies of analogues of the Rogers–Ramanujan functions. We likewise employ some of the relations proved in this work in order to derive partition-theoretic theorems.

The organization of this thesis is as follows. In Chapter 2, we study squares of the Rogers–Ramanujan functions and the parameters k, μ, and ν. Cubes of the Rogers–Ramanujan functions and connections to the Rogers–Ramanujan continued fraction are studied in Chapter 3, along with analogues for the Göllnitz–Gordon functions and the Göllnitz–Gordon continued fraction. Applications to the theory of partitions are given. In Chapter 4, we prove new relations for fourth and fifth powers of the Rogers–Ramanujan functions and offer applications to the Rogers–Ramanujan continued fraction. Chapter 5 is devoted to proving identities for the Rogers–Ramanujan functions conjectured by Koike and Somos. In Chapter 6, we give new proofs of identities for the Göllnitz-Gordon functions as well as for dodecic analogues of the Rogers–Ramanujan functions. Sextodecic analogues of the Rogers–Ramanujan functions are considered in Chapter 7.

Lastly, in Chapter 8, we give a simple proof of a classical theta-function inversion formula. In the proof, we use only basic knowledge in calculus. The proof is motivated by Berndt and K. Venkatachaliengerâs paper [22], which gave a simple proof of the corresponding transformation formula for the Dedekind-eta function, as well as work of S. Kongsiriwong [64] on a similar transformation formula.

Chapter 2

Two Modular Equations for Squares of the Rogers–Ramanujan Functions

2.1 Introduction

Recall the Rogers–Ramanujan continued fraction $R(q)$,

$$R(q) := \frac{q^{1/5}}{1} \, \frac{}{+} \, \frac{q}{1} \, \frac{}{+} \, \frac{q^2}{1} \, \frac{}{+} \, \frac{q^3}{1} \, \frac{}{+} \cdots, \qquad |q| < 1. \tag{2.1.1}$$

Recall also the Rogers–Ramanujan functions

$$G(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{f(-q^2,-q^3)}{f(-q)} \tag{2.1.2}$$

and

$$H(q) := \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q;q)_n} = \frac{f(-q,-q^4)}{f(-q)}. \tag{2.1.3}$$

In his important paper [81], Rogers proved the Rogers–Ramanujan identities, the second equality in each of (2.1.2) and (2.1.3). In that same paper, Rogers connected the Rogers–Ramanujan functions and the Rogers–Ramanujan continued fraction by proving that

$$R(q) = \frac{q^{1/5} f(-q,-q^4)}{f(-q^2,-q^3)} = \frac{q^{1/5} H(q)}{G(q)}. \tag{2.1.4}$$

In his Ph.D. thesis [80], Robins used the theory of modular forms to discover and prove new relations for $G(q)$ and $H(q)$. In particular, Robins found that

$$G^2(q)H(q^2) - H^2(q)G(q^2) = 2qH(q)H^2(q^2)\frac{f_{10}^2}{f_5^2} \tag{2.1.5}$$

and

$$G^2(q)H(q^2) + H^2(q)G(q^2) = 2G(q)G^2(q^2)\frac{f_{10}^2}{f_5^2}. \tag{2.1.6}$$

These identities were independently discovered by Gordon and McIntosh [46, Equations (3.10) and (3.11)]. Gordon and McIntosh gave elementary proofs of (2.1.5) and (2.1.6), and applied them in the context of studying mock theta functions. In Section 2.3, we prove new analogues of (2.1.5) and (2.1.6), and in

Section 2.5 we give new proofs of (2.1.5) and (2.1.6).

Observe that by dividing (2.1.5) by (2.1.6) and then using (2.1.4) and (1.0.27), we find that

$$\frac{G^2(q)H(q^2)-H^2(q)G(q^2)}{G^2(q)H(q^2)+H^2(q)G(q^2)}=q\frac{H(q)H^2(q^2)}{G(q)G^2(q^2)}=R(q)R^2(q^2)=k. \tag{2.1.7}$$

Since the identities (2.1.5) and (2.1.6) give us alternative representations of both the numerator and denominator of the fraction on the far left-side of (2.1.7), we have a powerful tool for proving results about k. Similar connections will be developed and utilized for μ and ν.

The organization of this chapter is as follows. In Section 2.2, we provide our first proof of the key identities (2.1.5) and (2.1.6). A second proof of these identities is presented in Section 2.5. In Section 2.3, we apply the identities (2.1.5), (2.1.6), and (2.1.7) to prove several results of Ramanujan involving k and $R(q)$. Most notably, we give a new proof of one of the most important formulas for $R(q)$ [78, pp. 135–177, 238–243], [97], [98], [14, pp. 265–267], [54], [7, p. 11], namely,

$$\frac{1}{R^5(q)}-11-R^5(q)=\frac{f^6(-q)}{qf^6(-q^5)}. \tag{2.1.8}$$

In Section 2.4, we apply our results of Section 2.3 to prove several theta function identities recorded in Ramanujan's notebooks. In Section 2.5, we prove identities for the Rogers–Ramanujan functions and connect them with the parameters μ and ν. We then establish for these parameters results that are analogous to those for k. Some of these results are new. As applications, we give new proofs for modular identities for $R(q)$ at the arguments q, $-q$, q^2, and q^4, and further offer new relations for $R(q)$ at these arguments.

2.2 First Proofs of (2.1.5), (2.1.6)

Our first proofs of (2.1.5) and (2.1.6) relies on (1.0.16) and (1.0.17). In Theorem 2.5.11, we present an entirely new proof of (2.1.5) and (2.1.6).

Proof of (2.1.5). Employing (2.1.2) and (2.1.3), we write (2.1.5) in the equivalent form

$$\frac{f^2(-q^2,-q^3)f(-q^2,-q^8)}{f^2(-q)f(-q^2)}-\frac{f^2(-q,-q^4)f(-q^4,-q^6)}{f^2(-q)f(-q^2)} \\ =2q\frac{f^2(-q^{10})}{f^2(-q^5)}\cdot\frac{f(-q,-q^4)f^2(-q^2,-q^8)}{f(-q)f^2(-q^2)}. \tag{2.2.1}$$

Multiplying (2.2.1) by

$$\frac{f^2(-q)f(-q^2)}{f(-q^2,-q^3)f(-q,-q^4)f(-q^2,-q^8)},$$

we deduce that (2.1.5) is equivalent to

$$\frac{f(-q^2,-q^3)}{f(-q,-q^4)} - \frac{f(-q,-q^4)f(-q^4,-q^6)}{f(-q^2,-q^8)f(-q^2,-q^3)} = 2q\frac{f(-q^2,-q^8)}{f(-q^2,-q^3)}\frac{f(-q)}{f(-q^2)}\frac{f^2(-q^{10})}{f^2(-q^5)}. \tag{2.2.2}$$

We now require the following identities, each of which is readily verified by the Jacobi Triple Product Identity (1.0.6):

$$\frac{f(-q,-q^4)f(-q^4,-q^6)}{f(-q^2,-q^8)f(-q^2,-q^3)} = \frac{f(q^2,q^3)}{f(q,q^4)} \tag{2.2.3}$$

and

$$\frac{f(-q^2,-q^8)}{f(-q^2,-q^3)} = \frac{f(-q,-q^9)f(-q^2)}{f(-q)f(-q^{10})}. \tag{2.2.4}$$

Employing (2.2.3) and (2.2.4) in (2.2.2), we find that (2.1.5) is equivalent to

$$\frac{f(-q^2,-q^3)}{f(-q,-q^4)} - \frac{f(q^2,q^3)}{f(q,q^4)} = 2qf(-q,-q^9)\frac{f(-q^{10})}{f^2(-q^5)}. \tag{2.2.5}$$

Employing the Jacobi Triple Product Identity (1.0.6) once more, we find that

$$f(-q,-q^4)f(q,q^4) = f(-q^2,-q^8)\frac{f^2(-q^5)}{f(-q^{10})}. \tag{2.2.6}$$

Adding the fractions on the left-hand side of (2.2.5) and utilizing (2.2.6), we arrive at

$$\frac{f(-q^2,-q^3)f(q,q^4) - f(q^2,q^3)f(-q,-q^4)}{f(-q^2,-q^8)} = 2qf(-q,-q^9). \tag{2.2.7}$$

Thus, to prove (2.1.5), it suffices to prove (2.2.7). The truth of (2.2.7) is clear, however, upon setting $a = q$, $b = q^4$, $c = -q^2$, $d = -q^3$ in (1.0.17). This completes the proof of (2.1.5). $\square$

Proof of (2.1.6). The proof of (2.1.6) is similar to the proof of (2.1.5). Utilizing (2.1.2) and (2.1.3), we write (2.1.6) in the equivalent form

$$\begin{aligned}\frac{f^2(-q^2,-q^3)f(-q^2,-q^8)}{f^2(-q)f(-q^2)} &+ \frac{f^2(-q,-q^4)f(-q^4,-q^6)}{f^2(-q)f(-q^2)} \\ &= 2\frac{f^2(-q^{10})}{f^2(-q^5)} \cdot \frac{f(-q^2,-q^3)f^2(-q^4,-q^6)}{f(-q)f^2(-q^2)}.\end{aligned} \tag{2.2.8}$$

Multiplying (2.2.8) by

$$\frac{f^2(-q)f(-q^2)}{f(-q^2,-q^3)f(-q,-q^4)f(-q^4,-q^6)}, \tag{2.2.9}$$

we deduce that (2.1.6) is equivalent to

$$\frac{f(-q^2,-q^3)f(-q^2,-q^8)}{f(-q,-q^4)f(-q^4,-q^6)} + \frac{f(-q,-q^4)}{f(-q^2,-q^3)} = 2\frac{f(-q^4,-q^6)}{f(-q^1,-q^4)}\frac{f(-q)}{f(-q^2)}\frac{f^2(-q^{10})}{f^2(-q^5)}. \tag{2.2.10}$$

We next utilize the following identities, each of which is readily verified by the Jacobi Triple Product Identity (1.0.6):

$$\frac{f(-q^2,-q^3)f(-q^2,-q^8)}{f(-q,-q^4)f(-q^4,-q^6)} = \frac{f(q,q^4)}{f(q^2,q^3)} \tag{2.2.11}$$

and

$$\frac{f(-q^4,-q^6)}{f(-q,-q^4)} = \frac{f(-q^3,-q^7)f(-q^2)}{f(-q)f(-q^{10})}. \tag{2.2.12}$$

Employing (2.2.11) and (2.2.12) in (2.2.10), we find that (2.1.6) is equivalent to

$$\frac{f(q,q^4)}{f(q^2,q^3)} + \frac{f(-q,-q^4)}{f(-q^2,-q^3)} = 2f(-q^3,-q^7)\frac{f(-q^{10})}{f^2(-q^5)}. \tag{2.2.13}$$

We employ (1.0.6) once more to deduce that

$$f(-q^2,-q^3)f(q^2,q^3) = f(-q^4,-q^6)\frac{f^2(-q^5)}{f(-q^{10})}. \tag{2.2.14}$$

Adding the fractions on the left-hand side of (2.2.13) and utilizing (2.2.14), we arrive at

$$\frac{f(q,q^4)f(-q^2,-q^3)+f(-q,-q^4)f(q^2,q^3)}{f(-q^4,-q^6)} = 2f(-q^3,-q^7). \tag{2.2.15}$$

Thus, (2.1.6) is equivalent to (2.2.15). Setting $a=q$, $b=q^4$, $c=-q^2$, $d=-q^3$ in (1.0.16), we readily verify the equality (2.2.15). This completes the proof of (2.1.6). $\square$

2.3 Applications of (2.1.5), (2.1.6), and (2.1.7)

Theorem 2.3.1. [16, Entry 1, pp. 12–13] *Let*

$$u := R(q)$$

and

$$v := R(q^2).$$

Then

(i) $$\frac{v-u^2}{v+u^2} = uv^2,$$

(ii) $$u^5 = k\left(\frac{1-k}{1+k}\right)^2 \qquad \text{and} \qquad v^5 = k^2\left(\frac{1+k}{1-k}\right),$$

(iii) $$\frac{1}{uv^2} - uv^2 = \frac{\chi^5(-q^5)}{q\chi(-q)}.$$

The modular equation in (i) was first established by Rogers [82, Eq. (5.4)], and later by N.J. Fine [41, p. 110, Eq. (40.48)]. A third proof was discovered by Andrews, Berndt, Jacobsen, and Lamphere [8, Entry 24]. In [17], Berndt, H.H. Chan, and S.–S. Huang applied (i) in order to establish an integral evaluation found in Ramanujan's notebooks. The identities in (ii) were first proved by Andrews, Berndt, Jacobsen, and Lamphere [8, Entry 24]. The identity in (iii) is new.

Proof of (i). We begin with (2.1.7), namely

$$\frac{G^2(q)H(q^2)-H^2(q)G(q^2)}{G^2(q)H(q^2)+H^2(q)G(q^2)}=q\frac{H(q)H^2(q^2)}{G(q)G^2(q^2)}. \tag{2.3.1}$$

Dividing all terms on the left-hand side of (2.3.1) by $G^2(q)G(q^2)$ and applying (2.1.4) and the definitions of u and v, we readily deduce (i). □

Proof of (ii). Use (2.1.7) to write

$$k\left(\frac{1-k}{1+k}\right)^2=R(q)R^2(q^2)\frac{\left(1-\dfrac{G^2(q)H(q^2)-H^2(q)G(q^2)}{G^2(q)H(q^2)+H^2(q)G(q^2)}\right)^2}{\left(1+\dfrac{G^2(q)H(q^2)-H^2(q)G(q^2)}{G^2(q)H(q^2)+H^2(q)G(q^2)}\right)^2}. \tag{2.3.2}$$

Simplifying the numerator and denominator on the right-hand side of (2.3.2) and then applying (2.1.4), we obtain

$$\begin{aligned}k\left(\frac{1-k}{1+k}\right)^2&=R(q)R^2(q^2)\left(\frac{2H^2(q)G(q^2)}{2G^2(q)H(q^2)}\right)^2\\&=R(q)R^2(q^2)\left(\frac{R^2(q)}{R(q^2)}\right)^2\\&=R^5(q).\end{aligned} \tag{2.3.3}$$

This proves the first equality in (ii). The proof of the second equality is similar, and we omit the details. □

Proof of (iii). Using (2.1.7) and simplifying, we find that

$$\begin{aligned}\frac{1}{uv^2}-uv^2&=\frac{G^2(q)H(q^2)+H^2(q)G(q^2)}{G^2(q)H(q^2)-H^2(q)G(q^2)}-\frac{G^2(q)H(q^2)-H^2(q)G(q^2)}{G^2(q)H(q^2)+H^2(q)G(q^2)}\\&=\frac{\left(G^2(q)H(q^2)+H^2(q)G(q^2)\right)^2-\left(G^2(q)H(q^2)-H^2(q)G(q^2)\right)^2}{\left(G^2(q)H(q^2)-H^2(q)G(q^2)\right)\left(G^2(q)H(q^2)+H^2(q)G(q^2)\right)}\\&=\frac{4G^2(q)H^2(q)G(q^2)H(q^2)}{\left(G^2(q)H(q^2)-H^2(q)G(q^2)\right)\left(G^2(q)H(q^2)+H^2(q)G(q^2)\right)}.\end{aligned} \tag{2.3.4}$$

Next, employ (2.1.5) and (2.1.6) to rewrite (2.3.4) as

$$\frac{1}{uv^2}-uv^2=\frac{4G^2(q)H^2(q)G(q^2)H(q^2)}{\left(2qH(q)H^2(q^2)\left(\frac{f_{10}}{f_5}\right)^2\right)\left(2G(q)G^2(q^2)\left(\frac{f_{10}}{f_5}\right)^2\right)}. \tag{2.3.5}$$

By the Jacobi Triple Product Identity (1.0.6), it is easy to show that

$$G(q)H(q)=\frac{f(-q^5)}{f(-q)}=\frac{f_5}{f_1}. \tag{2.3.6}$$

Employing (2.3.6) and (1.0.12) in (2.3.5), we deduce Theorem 2.3.1(iii). □

In his Lost Notebook, Ramanujan provided elegant parameterizations of certain quotients of theta functions in terms of the parameter k. In Theorems 2.3.2 and 2.3.4, we give proofs of five such parameterizations. Four of these parameterizations (Theorem 2.3.2(ii), (iii); Theorem 2.3.4(i), (ii)) were given by Ramanujan, and the fifth (Theorem 2.3.2(i)) is new. We remark that previously the only known proofs of Ramanujan's results were due to Kang [59, Theorem 4.2, Theorem 4.5], [7, Entry 1.8.2, p. 35; Entry 1.8.5, pp. 37–39]. We then show how several of these results can be used to establish further identities of Ramanujan.

Theorem 2.3.2. [7, Entry 1.8.2, p. 35], [59, Theorem 4.2] *If* $k \leq \sqrt{5}-2$, *then*

(i) $$q\frac{\chi(-q)}{\chi^5(-q^5)} = \frac{k}{1-k^2},$$

(ii) $$\frac{\phi^2(-q)}{\phi^2(-q^5)} = \frac{1-4k-k^2}{1-k^2},$$

(iii) $$\frac{\psi^2(q)}{q\psi^2(q^5)} = \frac{1+k-k^2}{k}.$$

We remark that the condition $k \leq \sqrt{5}-2$ is set to ensure that $1-4k-k^2 \geq 0$. Before we prove Theorem 2.3.2, we state two results of Ramanujan that we shall require.

Theorem 2.3.3. [14, Entry 9(ii), p. 258; Entry 10(v), p. 262] *We have*

(i) $$4q\frac{f^5(q^5)}{f(q)} + \frac{\phi^5(q^5)}{\phi(q)} = \phi(q)\phi^3(q^5),$$

(ii) $$\psi^2(q) - q\psi^2(q^5) = f(q,q^4)f(q^2,q^3).$$

Proof of Theorem 2.3.2(i). Using the definition of k, we see that Theorem 2.3.2(i) is simply a restatement of Theorem 2.3.1(iii). □

Proof of Theorem 2.3.2(ii). By Theorem 2.3.2(i),

$$\frac{1-4k-k^2}{1-k^2} = 1-4\frac{k}{1-k^2} = 1-4q\frac{\chi(-q)}{\chi^5(-q^5)}. \tag{2.3.7}$$

Hence, Theorem 2.3.2(ii) is equivalent to the theta function identity

$$\frac{\phi^2(-q)}{\phi^2(-q^5)} = 1-4q\frac{\chi(-q)}{\chi^5(-q^5)}. \tag{2.3.8}$$

By [14, Entry 24(iii), p. 39], we know that

$$\chi(q) = \frac{\phi(q)}{f(q)}. \tag{2.3.9}$$

Rearranging Theorem 2.3.3(i) and then applying (2.3.9), we find that

$$\begin{aligned}\frac{\phi^2(q)}{\phi^2(q^5)} - 1 &= 4q\frac{\phi(q)}{f(q)}\cdot\frac{f^5(q^5)}{\phi^5(q^5)} \\ &= 4q\chi(q)\cdot\frac{1}{\chi^5(q^5)}.\end{aligned} \tag{2.3.10}$$

Replacing q by $-q$ in (2.3.10), we easily arrive at (2.3.8). This completes the proof of (ii). $\square$

Proof of Theorem 2.3.2(iii). The proof of (iii) is similar to the proof of (ii). Using Theorem 2.3.2(i), we find that (iii) is equivalent to

$$\frac{\psi^2(q)}{\psi^2(q^5)} = q + \frac{\chi^5(-q^5)}{\chi(-q)}. \tag{2.3.11}$$

We show that (2.3.11) follows from Ramanujan's result, Theorem 2.3.3(ii). Dividing all terms of Theorem 2.3.3(ii) by $\psi^2(q^5)$, we obtain

$$\frac{\psi^2(q)}{\psi^2(q^5)} - q = \frac{f(q,q^4)f(q^2,q^3)}{\psi^2(q^5)}. \tag{2.3.12}$$

Employing (1.0.8) and the Jacobi Triple Product Identity (1.0.6), and then simplifying the resulting q-products, we find that

$$\begin{aligned}
\frac{f(q,q^4)f(q^2,q^3)}{\psi^2(q^5)} &= (-q;q^5)_\infty(-q^4;q^5)_\infty(q^5;q^5)_\infty \\
&\quad \times (-q^2;q^5)_\infty(-q^3;q^5)_\infty(q^5;q^5)_\infty \frac{(q^5;q^{10})^2_\infty}{(q^{10};q^{10})^2_\infty} \\
&= \frac{(-q;q)_\infty}{(-q^5;q^5)_\infty}\frac{(q^5;q^5)^2_\infty(q^5;q^{10})^2_\infty}{(q^{10};q^{10})^2_\infty} \\
&= (-q;q)_\infty \frac{(q^5;q^5)^3_\infty}{(q^{10};q^{10})^3_\infty}(q^5;q^{10})^2_\infty \\
&= (-q;q)_\infty (q^5;q^{10})^3_\infty (q^5;q^{10})^2_\infty \\
&= \frac{(q^5;q^{10})^5_\infty}{(q;q^2)_\infty} \\
&= \frac{\chi^5(-q^5)}{\chi(-q)},
\end{aligned}$$

where in the penultimate line we applied Euler's Identity [14, Equation (22.3), p. 37],

$$(-q;q)_\infty = \frac{1}{(q;q^2)_\infty}, \tag{2.3.13}$$

and in the last line we invoked (1.0.10). This establishes (2.3.11), and completes the proof of (iii). $\square$

In the next theorem, we establish two more parameterizations of Ramanujan.

Theorem 2.3.4. [7, Entry 1.8.5, p. 37], [59, Theorem 4.5] *If* $k \leq \sqrt{5}-2$, *then*

(i) $$\frac{k}{1-k^2}\left(\frac{1+k-k^2}{1-4k-k^2}\right)^5 = q(-q;q)^{24}_\infty,$$

(ii) $$\left(\frac{k}{1-k^2}\right)^5 \frac{1+k-k^2}{1-4k-k^2} = q^5(-q^5;q^5)^{24}_\infty.$$

We include a remark on this theorem given in [7, pp. 37–38], [59, pp. 104–

105]. Let $\Delta(\tau)$ denote the discriminant function defined by

$$\Delta(\tau) = q(q;q)_\infty^{24},$$

where $q = e^{2\pi i\tau}$ and $\operatorname{Im} \tau > 0$. Using the definition of Δ, we can easily see that the identities in Theorem 2.3.4 are representations of certain quotients of Δ–functions in terms of k, namely,

$$\frac{k}{1-k^2}\left(\frac{1+k-k^2}{1-4k-k^2}\right)^5 = \frac{\Delta(2\tau)}{\Delta(\tau)}$$

and

$$\left(\frac{k}{1-k^2}\right)^5 \frac{1+k-k^2}{1-4k-k^2} = \frac{\Delta(10\tau)}{\Delta(5\tau)},$$

respectively.

Proof of Theorem 2.3.4. Apply Theorem 2.3.2 and (1.0.12) to deduce that

$$\begin{aligned}
\frac{k}{1-k^2}\left(\frac{1+k-k^2}{1-4k-k^2}\right)^5 &= \frac{k^6}{(1-k^2)^6}\left(\frac{1+k-k^2}{k}\right)^5\left(\frac{1-k^2}{1-4k-k^2}\right)^5 \\
&= \frac{q^6\chi^6(-q)}{\chi^{30}(-q^5)}\cdot\frac{\psi^{10}(q)}{q^5\psi^{10}(q^5)}\cdot\frac{\phi^{10}(-q^5)}{\phi^{10}(-q)}. \\
&= q\frac{f_1^6}{f_2^6}\cdot\frac{f_{10}^{30}}{f_5^{30}}\cdot\frac{f_2^{10}}{f_1^{20}}\cdot\frac{f_5^{20}}{f_{10}^{10}}\cdot\frac{f_5^{10}}{f_{10}^{20}}\cdot\frac{f_2^{20}}{f_1^{10}} \\
&= q\frac{f_2^{24}}{f_1^{24}} \\
&= q\frac{(q^2;q^2)_\infty^{24}}{(q;q)_\infty^{24}} \\
&= q(-q;q)_\infty^{24}.
\end{aligned}$$

This completes the proof of (i). The proof of (ii) is similar. We omit the details. $\square$

Corollary 2.3.5.

(i) $$\chi^4(-q)\chi^4(-q^5) = q\frac{\left(1-k^2\right)\left(1-4k-k^2\right)}{k\left(1+k-k^2\right)},$$

(ii) $$\frac{\chi^6(-q)}{\chi^6(-q^5)} = \frac{1}{q}\frac{k(1-4k-k^2)}{(1-k^2)(1+k-k^2)},$$

(iii) $$\frac{\chi^5(-q)}{\chi(-q^5)} = \frac{1-4k-k^2}{1+k-k^2}.$$

Proof. Multiply together the reciprocals of Theorem 2.3.4(i) and (ii) to obtain

$$\frac{1}{q^6(-q;q)_\infty^{24}(-q^5;q^5)_\infty^{24}} = \frac{\left(1-k^2\right)^6\left(1-4k-k^2\right)^6}{k^6\left(1+k-k^2\right)^6}. \tag{2.3.14}$$

Apply (2.3.13) and (1.0.10) on the left-hand side of (2.3.14), multiply the resulting equation by q^6, and then take sixth roots of both sides to obtain (i).

The proof of (ii) is similar, and follows from multiplying Theorem 2.3.4(ii) by the reciprocal of Theorem 2.3.4(i), applying (2.3.13) and (1.0.10), and taking fourth roots.

Part (iii) follows from multiplying parts (i) and (ii) and taking square roots of both sides. □

In the following theorem, we derive some simple consequences of (2.1.5), (2.1.6), and Theorem 2.3.1(i). The identities provide further relations involving squares of the Rogers–Ramanujan functions. The equality in (v) was discovered empirically by Michael Somos [93] on a computer; we provide the first proof of this identity.

Theorem 2.3.6. *The following relations hold:*

$$\text{(i)} \qquad G^2(q^2)G(q) + qH^2(q^2)H(q) = G^2(q)H(q^2)\frac{f_5^2}{f_{10}^2},$$

$$\text{(ii)} \qquad G^2(q^2)G(q) - qH^2(q^2)H(q) = H^2(q)G(q^2)\frac{f_5^2}{f_{10}^2},$$

$$\text{(iii)} \qquad \begin{aligned}&(G^2(q^2)G(q) + qH^2(q^2)H(q))\\ &\times(G^2(q^2)G(q) - qH^2(q^2)H(q)) = \frac{f_5^6}{f_1^2 f_2 f_{10}^3},\end{aligned}$$

$$\text{(iv)} \qquad \frac{G^2(q^2)G(q) - qH^2(q^2)H(q)}{G^2(q^2)G(q) + qH^2(q^2)H(q)} = \frac{u^2}{v},$$

$$\text{(v)} \qquad \begin{aligned}&H(q^2)G^2(q^2)G^3(q) - G(q)H^2(q)G^3(q^2)\\ &\qquad = qH(q)G^2(q)H^3(q^2) + qG(q^2)H^2(q^2)H^3(q).\end{aligned}$$

Proof of (i), (ii). Add (2.1.5) to (respectively subtract (2.1.5) from) (2.1.6) and simplify in order to derive (i) (respectively (ii)). □

Proof of (iii). Multiply together (i) and (ii) and simplify with the help of (2.3.6). □

Proof of (iv). Divide (ii) by (i) and simplify with the help of (2.1.4). □

Proof of (v). Rewrite Theorem 2.3.1(i) in the form

$$v - u^2 = uv^3 + u^3v^2. \tag{2.3.15}$$

Applying (2.1.4), we write (2.3.15) as

$$q^{2/5}\left[\frac{H(q^2)}{G(q^2)} - \frac{H^2(q)}{G^2(q)}\right] = q^{7/5}\frac{H(q)}{G(q)}\cdot\frac{H^3(q^2)}{G^3(q^2)} + q^{7/5}\frac{H^3(q)}{G^3(q)}\cdot\frac{H^2(q^2)}{G^2(q^2)}. \tag{2.3.16}$$

Multiplying (2.3.16) by $q^{-2/5}G^3(q)G^3(q^2)$ and rearranging the result, we complete the proof. □

Next, we derive new analogues of (2.1.5), (2.1.6), and Theorem 2.3.1(i). Define

$$\alpha := \frac{\psi^2(q)}{q\psi^2(q^5)}\left[\frac{\psi^2(q)}{q\psi^2(q^5)} - 2 + 5\frac{q\psi^2(q^5)}{\psi^2(q)}\right]. \tag{2.3.17}$$

Theorem 2.3.7. *Let*

$$u = R(q) \qquad \textit{and} \qquad v = R(q^2).$$

Then,

(i) $$G^4(q)H^2(q^2) - H^4(q)G^2(q^2) = 4kG^2(q)G^4(q^2)\frac{f_{10}^4}{f_5^4},$$

(ii) $$G^4(q)H^2(q^2) + H^4(q)G^2(q^2) = 2\sqrt{\alpha}kG^2(q)G^4(q^2)\frac{f_{10}^4}{f_5^4},$$

(iii) $$\frac{v^2+u^4}{v^2-u^4} = \frac{1}{2}\sqrt{\alpha}.$$

Proof of (i). Part (i) is immediate upon multiplying (2.1.5) with (2.1.6) and using (2.1.7). □

Proof of (ii). By Theorem 2.3.1(ii),

$$\begin{aligned} v^2 + u^4 &= k^{4/5}\left(\frac{1+k}{1-k}\right)^{2/5} + k^{4/5}\left(\frac{1-k}{1+k}\right)^{8/5} \\ &= \frac{2k^{4/5}k}{(1-k)^{2/5}(1+k)^{8/5}}\left(\frac{1+k^2}{k}\right). \end{aligned} \tag{2.3.18}$$

Applying (2.1.7) and (2.1.6), we find that

$$\begin{aligned} \frac{k^{4/5}}{(1-k)^{2/5}(1+k)^{8/5}} &= k^{4/5}\left(1 - \frac{G^2(q)H(q^2) - H^2(q)G(q^2)}{G^2(q)H(q^2) + H^2(q)G(q^2)}\right)^{-2/5} \\ &\quad \times \left(1 + \frac{G^2(q)H(q^2) - H^2(q)G(q^2)}{G^2(q)H(q^2) + H^2(q)G(q^2)}\right)^{-8/5} \\ &= k^{4/5}\frac{\left(G^2(q)H(q^2) + H^2(q)G(q^2)\right)^2}{\left(2H^2(q)G(q^2)\right)^{2/5}\left(2G^2(q)H(q^2)\right)^{8/5}} \\ &= \frac{q^{4/5}H^{4/5}(q)H^{8/5}(q^2)}{G^{4/5}(q)G^{8/5}(q^2)} \\ &\quad \times \frac{4G^2(q)G^4(q^2)\frac{f_{10}^4}{f_5^4}}{\left(2H^2(q)G(q^2)\right)^{2/5}\left(2G^2(q)H(q^2)\right)^{8/5}} \\ &= \frac{q^{4/5}G^2(q^2)}{G^2(q)}\frac{f_{10}^4}{f_5^4}. \end{aligned} \tag{2.3.19}$$

With the use of Theorem 2.3.2(iii), we can show that

$$\frac{1+k^2}{k} = \sqrt{\alpha}. \tag{2.3.20}$$

Substituting (2.3.19) and (2.3.20) into (2.3.18), we conclude that

$$v^2 + u^4 = 2q^{4/5}k\sqrt{\alpha}\frac{G^2(q^2)}{G^2(q)}\frac{f_{10}^4}{f_5^4}. \tag{2.3.21}$$

Using (2.1.4) and multiplying (2.3.21) by $q^{-4/5}G^4(q)G^2(q^2)$, we deduce (ii). □

Proof of (iii). Part (iii) follows from taking the quotient of parts (i) and (ii), and simplifying as in the proof of Theorem 2.3.1(i). $\square$

Now we show that, as a consequence of our work above, we have the following famous and fundamental result of Ramanujan.

Theorem 2.3.8. [78, pp. 135–177, 238–243], [97], [98], [14, pp. 265–267], [7, p. 11]

$$\frac{1}{R^5(q)} - 11 - R^5(q) = \frac{f^6(-q)}{qf^6(-q^5)}. \tag{2.3.22}$$

This equality was found in Ramanujan's notebooks by G.N. Watson [97], [98] and proved by Watson [97] in order to establish claims of Ramanujan about the Rogers–Ramanujan continued fraction. A different proof of (2.3.22) can be found in Berndt's book [14, pp. 265–267]. The identity (2.3.22) can also be found in an unpublished manuscript of Ramanujan first appearing in handwritten form with his lost notebook [78, pp. 135–177, 238–243].

Proof. We begin with [14, Entry 24(ii), p. 39], namely,

$$f^3(-q) = \phi^2(-q)\psi(q). \tag{2.3.23}$$

Employing (2.3.23), Theorem 2.3.2(ii), (iii), rearranging algebraically, and finally applying Theorem 2.3.1(ii), we deduce that

$$\begin{aligned}
\frac{f^6(-q)}{qf^6(-q^5)} &= \frac{\psi^2(q)}{q\psi^2(q^5)} \cdot \frac{\phi^4(-q)}{\phi^4(-q^5)} \\
&= \left(\frac{1+k-k^2}{k}\right)\left(\frac{1-4k-k^2}{1-k^2}\right)^2 \\
&= \frac{1}{k}\left(\frac{1+k}{1-k}\right)^2 - 11 - k\left(\frac{1-k}{1+k}\right)^2 \\
&= \frac{1}{R^5(q)} - 11 - R^5(q). \qquad (2.3.24)
\end{aligned}$$

This completes the proof. $\square$

It is well known ([67, p. 408], [19, p. 7]) and easy to show that, with the use of (2.3.6) and (2.1.4), identities (2.3.22) and (1.0.24) are equivalent. Hence, our new proof of (2.3.22) is also a new proof of (1.0.24). As noted in the Introduction, (1.0.24) is one of Ramanujan's identities for the Rogers–Ramanujan functions, and is also one of two identities stated by Ramanujan without proof in [74], [75, p. 231]. The first published proof of (1.0.24) was by H.B.C. Darling [39] in 1921. A second proof by Rogers [82] appeared in the same year. One year later, L.J. Mordell [67] found another proof.

2.4 Applications of k to Theta Function Identities of Degree Five

In the preceding section, applications of (2.1.5), (2.1.6), and (2.1.7) were made to prove representations of certain quotients of theta functions in terms of the parameter k; further applications were made to establishing modular relations for the Rogers–Ramanujan functions and the Rogers–Ramanujan continued fraction. In this section, we demonstrate how the results of the preceding section can be applied to give new proofs of certain theta function identities of degree five.

The following theorem of Ramanujan is in [14, Entry 4(i), p. 463]. We remark that this theta function identity has a remarkable representation in terms of Lambert series. Indeed, Ramanujan asserted that each side of (2.4.2) is equal to

$$1+6\sum_{k=1}^{\infty}\frac{kq^k}{1-q^k}-30\sum_{k=1}^{\infty}\frac{kq^{5k}}{1-q^{5k}}. \tag{2.4.1}$$

For proofs of this assertion, see [14, Entry 4(i), p. 463] and [65].

Theorem 2.4.1.

$$\begin{aligned}&\frac{\{f^{12}(-q)+22qf^6(-q)f^6(-q^5)+125q^2f^{12}(-q^5)\}^{1/2}}{f(-q)f(-q^5)}\\&=\frac{\psi^4(q)+2q\psi^2(q)\psi^2(q^5)+5q^2\psi^4(q^5)}{\psi(q)\psi(q^5)}\\&\quad\times\{\psi^4(q)-2q\psi^2(q)\psi^2(q^5)+5q^2\psi^4(q^5)\}^{1/2}.\end{aligned} \tag{2.4.2}$$

Proof. From (2.3.24), we have the identity

$$\frac{f^6(-q)}{qf^6(-q^5)}=\left(\frac{1+k-k^2}{k}\right)\left(\frac{1-4k-k^2}{1-k^2}\right)^2. \tag{2.4.3}$$

Employing (2.4.3), we rewrite the left-hand side of (2.4.2) as

$$\begin{aligned}&q^{1/2}f^2(-q)f^2(-q^5)\left[\frac{f^6(-q)}{qf^6(-q^5)}+22+125\frac{qf^6(-q^5)}{f^6(-q)}\right]^{1/2}\\&=q^{1/2}f^2(-q)f^2(-q^5)\left[\left(\frac{1+k-k^2}{k}\right)\left(\frac{1-4k-k^2}{1-k^2}\right)^2+22\right.\\&\qquad\left.+125\left(\frac{1-k^2}{1-4k-k^2}\right)^2\left(\frac{k}{1+k-k^2}\right)\right]^{1/2}\\&=q^{1/2}f^2(-q)f^2(-q^5)\left[\frac{\left(1+k^2\right)^2\left(1+4k+6k^2-4k^3+k^4\right)^2}{k\left(1-k^2\right)^2\left(1-4k-k^2\right)^2\left(1+k-k^2\right)}\right]^{1/2}.\end{aligned} \tag{2.4.4}$$

Similarly, with the help of Theorem 2.3.2(iii), we write the right-hand side of (2.4.2) as

$$\begin{aligned}&q^{3/2}\psi^2(q)\psi^2(q^5)\left[\frac{\psi^2(q)}{q\psi^2(q^5)}+2+5\frac{q\psi^2(q^5)}{\psi^2(q)}\right]\times\left[\frac{\psi^2(q)}{q\psi^2(q^5)}-2+5\frac{q\psi^2(q^5)}{\psi^2(q)}\right]^{1/2}\\&=q^{3/2}\psi^2(q)\psi^2(q^5)\left[\frac{1+k-k^2}{k}+2+5\frac{k}{1+k-k^2}\right]\end{aligned}$$

$$\times\left[\frac{1+k-k^2}{k}-2+5\frac{k}{1+k-k^2}\right]^{1/2}$$

$$=q^{3/2}\psi^2(q)\psi^2(q^5)\left[\frac{1+4k+6k^2-4k^3+k^4}{k\left(1+k-k^2\right)}\right]\times\left[\frac{\left(1+k^2\right)^2}{k\left(1+k-k^2\right)}\right]^{1/2}. \quad (2.4.5)$$

Equating (2.4.4) and (2.4.5) and simplifying, we see that, in order to prove (2.4.2), it suffices to prove that

$$\frac{f^2(-q)f^2(-q^5)}{\psi^2(q)\psi^2(q^5)}=q\frac{\left(1-k^2\right)\left(1-4k-k^2\right)}{k\left(1+k-k^2\right)}. \quad (2.4.6)$$

Applying (1.0.12) to the left-hand side of (2.4.6) and Corollary 2.3.5(i) to the right-hand side of (2.4.6), we find that both sides of (2.4.6) equal

$$\chi^4(-q)\chi^4(-q^5).$$

This establishes (2.4.6), and completes the proof. $\square$

On page 56 of his Lost Notebook, Ramanujan recorded the following identities:

Theorem 2.4.2. [7, Entry 1.6.1, p. 26], [59, Theorem 2.1]

(i) $$\frac{f^3(-q)}{f^3(-q^5)}=\frac{\psi(q)}{\psi(q^5)}\times\frac{\psi^2(q)-5q\psi^2(q^5)}{\psi^2(q)-q\psi^2(q^5)},$$

(ii) $$\frac{f^6(-q^2)}{f^6(-q^{10})}=\frac{\psi^4(q)}{\psi^4(q^5)}\times\frac{\psi^2(q)-5q\psi^2(q^5)}{\psi^2(q)-q\psi^2(q^5)},$$

(iii) $$\frac{f^3(-q^2)}{qf^3(-q^{10})}=\frac{\varphi(q)}{\varphi(q^5)}\times\frac{5\varphi^2(q^5)-\varphi^2(q)}{\varphi^2(q)-\varphi^2(q^5)},$$

(iv) $$\frac{f^6(-q)}{qf^6(-q^5)}=\frac{\varphi^4(-q)}{\varphi^4(-q^5)}\times\frac{5\varphi^2(-q^5)-\varphi^2(-q)}{\varphi^2(-q^5)-\varphi^2(-q)}.$$

Proof of (i), (ii). Utilizing Theorem 2.3.2(iii), we deduce that

$$\begin{aligned}\frac{\psi^2(q)-5q\psi^2(q^5)}{\psi^2(q)-q\psi^2(q^5)}&=\frac{\dfrac{\psi^2(q)}{q\psi^2(q^5)}-5}{\dfrac{\psi^2(q)}{q\psi^2(q^5)}-1}\\&=\frac{\dfrac{1+k-k^2}{k}-5}{\dfrac{1+k-k^2}{k}-1}\\&=\frac{1-4k-k^2}{1-k^2}.\end{aligned} \quad (2.4.7)$$

Taking positive square roots in Theorem 2.3.2(iii), we also deduce that

$$\frac{\psi(q)}{\psi(q^5)}=\sqrt{q}\sqrt{\frac{1+k-k^2}{k}}. \quad (2.4.8)$$

From (2.4.3), we find upon taking positive square roots that

$$\frac{f^3(-q)}{f^3(-q^5)} = \sqrt{q}\sqrt{\frac{1+k-k^2}{k}}\left(\frac{1-4k-k^2}{1-k^2}\right). \tag{2.4.9}$$

Comparing (2.4.9) with the product of (2.4.7) and (2.4.8), we readily deduce the truth of (i). To deduce (ii), multiply (i) by $\psi^3(q)/\psi^3(q^5)$ and note that, by (1.0.12),

$$\frac{f^3(-q)}{f^3(-q^5)} \cdot \frac{\psi^3(q)}{\psi^3(q^5)} = \frac{f^6(-q^2)}{f^6(-q^{10})}. \tag{2.4.10}$$

□

Proof of (iii), (iv). The proof of (iv) is analogous to the proof of (i), but employs part (ii) in place of part (iii) of Theorem 2.3.2. The derivation of (iii) from (iv) is similar to the derivation of (ii) from (i). We omit the details. □

From Theorem 2.4.2 we extract another interesting representation for a quotient of theta functions in terms of k. It is interesting to compare this result with (2.4.3).

Corollary 2.4.3. *We have*

$$\frac{f^6(-q^2)}{q^2 f^6(-q^{10})} = \left(\frac{1-4k-k^2}{1-k^2}\right)\left(\frac{1+k-k^2}{k}\right)^2.$$

Proof. Apply (2.4.7) and Theorem 2.3.2(iii) to the right-hand side of Theorem 2.4.2(ii). □

The next pair of identities does not appear in the Lost Notebook. They were established by Kang, who employed them to prove certain factorization theorems claimed by Ramanujan [59].

Theorem 2.4.4. [7, Theorem 1.6.1, p. 27], [59, Theorem 2.2] *We have*

(i) $$\phi^2(q) - 5\phi^2(q^5) = -4f^2(-q^2)\frac{\chi(q^5)}{\chi(q)},$$

(ii) $$\psi^2(q) - 5q\psi^2(q^5) = f^2(-q)\frac{\chi(-q)}{\chi(-q^5)}.$$

Proof of (i). We prove the equivalent form

$$\frac{\phi^2(-q)}{\phi^2(-q^5)} - 5 = \frac{-4f^2(-q^2)\chi(-q^5)}{\phi^2(-q^5)\chi(-q)}, \tag{2.4.11}$$

which we obtain from (i) by dividing all terms by $\phi^2(q^5)$ and then replacing q by $-q$. Applying Theorem 2.3.2(ii), (i), (iii), we deduce that

$$\begin{aligned}\frac{\phi^2(-q)}{\phi^2(-q^5)} - 5 &= \frac{1-4k-k^2}{1-k^2} - 5\\ &= -4\frac{1+k-k^2}{k} \cdot \frac{k}{1-k^2}\\ &= -4\frac{\psi^2(q)}{q\psi^2(q^5)} \cdot \frac{q\chi(-q)}{\chi^5(-q^5)}.\end{aligned} \tag{2.4.12}$$

Applying (1.0.12), we readily conclude that

$$-4\frac{\psi^2(q)}{q\psi^2(q^5)}\cdot\frac{q\chi(-q)}{\chi^5(-q^5)}=-4\frac{f^2(-q^2)\chi(-q^5)}{\phi^2(-q^5)\chi(-q)},$$

which completes the proof of (i). ☐

Proof of (ii). The proof of (ii) is analogous to the proof of (i). ☐

On page 50 of the Lost Notebook, Ramanujan recorded the following pair of identities.

Theorem 2.4.5. [7, Entry 1.6.2, p. 28], [59, Corollary 2.3]

(i) $$16qf^2(-q^2)f^2(-q^{10})=\left(\phi^2(q)-\phi^2(q^5)\right)\left(5\phi^2(q^5)-\phi^2(q)\right),$$

(ii) $$f^2(-q)f^2(-q^5)=\left(\psi^2(q)-q\psi^2(q^5)\right)\left(\psi^2(q)-5q\psi^2(q^5)\right).$$

Proof. Rewrite (i) in the equivalent form

$$\frac{16qf^2(-q^2)f^2(-q^{10})}{\phi^4(-q^5)}=\left(\frac{\phi^2(-q)}{\phi^2(-q^5)}-1\right)\left(\frac{\phi^2(-q)}{\phi^2(-q^5)}-5\right), \tag{2.4.13}$$

and proceed as in the proof of Theorem 2.4.4. The proof of (ii) is analogous. ☐

2.5 Analogues of (2.1.5), (2.1.6), and (2.1.7) for μ and ν

Recall the definitions of μ and ν from (1.0.28) and (1.0.29), respectively. The parameters μ and ν are connected to certain identities for the Rogers–Ramanujan functions in analogy with the way that k is connected to the identities (2.1.5), (2.1.6) and (2.1.7). Specifically, we will need the following identities.

Theorem 2.5.1. *The following identities hold:*

(i) $$G(q)G(q^4)-qH(q)H(q^4)=\frac{\phi(q^5)}{f(-q^2)},$$

(ii) $$G(q)G(q^4)+qH(q)H(q^4)=\frac{\phi(q)}{f(-q^2)},$$

(iii) $$G(q)H(-q)-G(-q)H(q)=2q\frac{\psi(q^{10})}{f(-q^2)},$$

(iv) $$G(q)H(-q)+G(-q)H(q)=2\frac{\psi(q^2)}{f(-q^2)},$$

(v) $$G^2(q)G(q^2)H(q^4)-H^2(q)H(q^2)G(q^4)=2q\frac{\psi(q^{10})\psi(-q^5)}{f(-q^2)\psi(-q)},$$

(vi) $$G^2(q)G(q^2)H(q^4)+H^2(q)H(q^2)G(q^4)=2\frac{\psi(q^2)\psi(-q^5)}{f(-q^2)\psi(-q)}.$$

The first four identities are found among Ramanujan's list of 40 modular relations for the Rogers–Ramanujan functions [78, pp. 236–237], [19, pp. 7–11]. They are, respectively, Entry 3.3, 3.2, 3.21, and 3.20 in [19]. The fifth and sixth identities are remarkable equations relating the Rogers–Ramanujan functions at *three* different arguments. We will see below that the fifth and sixth identities are actually equivalent to the third and fourth identities, respectively, though they do not appear in the forms (v) and (vi) anywhere in Ramanujan's work. We will also show that the identities in Theorem 2.5.1 follow naturally from an entry of Ramanujan, which appears on page 56 of his Lost Notebook. This entry was first proved by Kang [7, Entry 1.7.1, p. 28], [59, Theorem 3.1] , and we record it as Theorem 2.5.2.

Theorem 2.5.2. [7, Entry 1.7.1, p. 28], [59, Theorem 3.1] *We have*

$$\text{(i)} \qquad \phi(q)+\phi(q^5)=2q^{4/5}f(q,q^9)R^{-1}(q^4),$$
$$\text{(ii)} \qquad \phi(q)-\phi(q^5)=2q^{1/5}f(q^3,q^7)R(q^4),$$
$$\text{(iii)} \qquad \psi(q^2)+q\psi(q^{10})=q^{1/5}f(q^2,q^8)R^{-1}(q),$$
$$\text{(iv)} \qquad \psi(q^2)-q\psi(q^{10})=q^{-1/5}f(q^4,q^6)R(q).$$

Proof of Theorem 2.5.1(i), (ii). Subtracting (ii) from (i) in Theorem 2.5.2, and using (2.1.4), we find that

$$\begin{aligned}\phi(q^5)&=q^{4/5}\frac{f(q,q^9)}{R(q^4)}-q^{1/5}f(q^3,q^7)R(q^4)\\&=\frac{f(q,q^9)f(-q^8,-q^{12})}{f(-q^4,-q^{16})}-q\frac{f(q^3,q^7)f(-q^4,-q^{16})}{f(-q^8,-q^{12})}.\end{aligned}\tag{2.5.1}$$

By three applications each of the Jacobi Triple Product Identity (1.0.6) and of (1.0.9),

$$\begin{aligned}\frac{f(q,q^9)}{f(-q^4,-q^{16})}&=\frac{(-q;q^{10})_\infty(-q^9;q^{10})_\infty(q^{10};q^{10})_\infty}{(q^4;q^{20})_\infty(q^{16};q^{20})_\infty(q^{20};q^{20})_\infty}\\&=\frac{(-q;q^{10})_\infty(-q^9;q^{10})_\infty}{(-q^2;q^{10})_\infty(q^2;q^{10})_\infty(-q^8;q^{10})_\infty(q^8;q^{10})_\infty(-q^{10};q^{10})_\infty}\\&=\frac{(-q;q^{10})_\infty(-q^9;q^{10})_\infty}{(-q^2;q^{10})_\infty(-q;q^5)_\infty(-q^8;q^{10})_\infty(-q^4;q^5)_\infty(-q^{10};q^{10})_\infty}\\&\qquad\times\frac{1}{(q;q^5)_\infty(q^4;q^5)_\infty}\\&=\frac{1}{(-q^2;q^2)_\infty}\times\frac{(q^2;q^5)_\infty(q^3;q^5)_\infty(q^5;q^5)_\infty}{(q;q)_\infty}\\&=\frac{(q^2;q^2)_\infty}{(q^4;q^4)_\infty}\times\frac{f(-q^2,-q^3)}{f(-q)}\\&=\frac{f(-q^2)}{f(-q^4)}\times\frac{f(-q^2,-q^3)}{f(-q)}.\end{aligned}\tag{2.5.2}$$

Hence, by (2.1.2),

$$\frac{f(q,q^9)f(-q^8,-q^{12})}{f(-q^2)f(-q^4,-q^{16})}=G(q)G(q^4).\tag{2.5.3}$$

By completely analogous product manipulations, we find that

$$\frac{f(q^3,q^7)f(-q^4,-q^{16})}{f(-q^2)f(-q^8,-q^{12})}=H(q)H(q^4).\tag{2.5.4}$$

Multiplying (2.5.1) by $1/f(-q^2)$ and substituting (2.5.3) and (2.5.4) into the resulting equation, we establish Theorem 2.5.1(i).

The proof of Theorem 2.5.1(ii) follows by adding Theorem 2.5.2(i) and (ii) and then using (2.5.3) and (2.5.4) as above. □

Proof of Theorem 2.5.1(iii), (iv). Using (1.0.6) and elementary product manipulations, we can show that

$$G(q)G(-q) = \frac{f(q^4, q^6)}{f(-q^2)} \qquad \text{and} \qquad H(q)H(-q) = \frac{f(q^2, q^8)}{f(-q^2)}. \tag{2.5.5}$$

Substituting (2.5.5) into Theorem 2.5.2(iii) and (iv), simplifying with the use of (2.1.4), and then subtracting (respectively adding) the resulting equations, we prove Theorem 2.5.1(iii) and (iv). □

Proof of Theorem 2.5.1(v), (vi). Use (1.0.13) and (2.3.6) to write

$$\begin{aligned} \frac{\psi(-q^5)}{\psi(-q)} &= \frac{f(-q^2)f(-q^5)f(-q^{20})}{f(-q)f(-q^4)f(-q^{10})} \\ &= \frac{G(q)H(q)G(q^4)H(q^4)}{G(q^2)H(q^2)}. \end{aligned} \tag{2.5.6}$$

Applying (1.0.6) and manipulating the resulting products, we can show that

$$G(-q) = \frac{G(q^2)H^2(q^2)}{G(q)H(q^4)} \qquad \text{and} \qquad H(-q) = \frac{G^2(q^2)H(q^2)}{G(q^4)H(q)}. \tag{2.5.7}$$

Multiplying Theorem 2.5.1(iii) by

$$\frac{G(q)H(q)G(q^4)H(q^4)}{G(q^2)H(q^2)}$$

and applying (2.5.6) and (2.5.7), we see that Theorem 2.5.1(iii) and (v) are equivalent. Similarly, we establish the equivalency of Theorem 2.5.1(iv) and (vi). □

Next, we apply Theorem 2.5.1 to establish analogues of Theorem 2.3.2, Theorem 2.3.4, and Corollary 2.3.5.

Theorem 2.5.3. *We have*

(i) $$\frac{\phi(q)}{\phi(q^5)} = \frac{1+\mu}{1-\mu},$$

(ii) $$\frac{q\chi(q)}{\chi^5(q^5)} = \frac{\mu}{(1-\mu)^2},$$

(iii) $$\frac{\psi^2(-q)}{q\psi^2(-q^5)} = \frac{1-3\mu+\mu^2}{\mu}.$$

Theorem 2.5.3(i) is stated on page 26 in the Lost Notebook, and has been proved by Kang [7, Entry 1.8.1(i) p. 33], [59, Theorem 4.1(i)]. The remaining representations are new.

Proof of Theorem 2.5.3(i). In Theorem 2.5.1, divide part (ii) by part (i) to obtain

$$\frac{\phi(q)}{\phi(q^5)} = \frac{G(q)G(q^4) + qH(q)H(q^4)}{G(q)G(q^4) - qH(q)H(q^4)}. \tag{2.5.8}$$

Divide the numerator and denominator on the right-hand side of (2.5.8) by $G(q)G(q^4)$ and use (2.1.4) and (1.0.28) to complete the proof. □

Proof of Theorem 2.5.3(ii). Use Theorem 2.5.3(i) in (2.3.10). □

Proof of Theorem 2.5.3(iii). Replace q by $-q$ in (2.3.11) and use Theorem 2.5.3 (ii) in the resulting equation. □

Theorem 2.5.4. [7, Theorem 1.8.1, p. 39], [59, Theorem 4.6] *If* $q = e^{2\pi i\tau}$, *where* $\operatorname{Im}\tau > 0$, *then*

$$\text{(i)} \qquad \frac{\mu}{(1-\mu)^2}\left(\frac{1-3\mu+\mu^2}{1+2\mu+\mu^2}\right)^5 = \frac{q}{(-q;q^2)_\infty^{24}} = -\frac{\Delta(2\tau)}{\Delta(1/2+\tau)},$$

$$\text{(ii)} \qquad \left(\frac{\mu}{(1-\mu)^2}\right)^5 \frac{1-3\mu+\mu^2}{1+2\mu+\mu^2} = \frac{q^5}{(-q^5;q^{10})_\infty^{24}} = -\frac{\Delta(10\tau)}{\Delta(1/2+5\tau)}.$$

Proof. Observe that by (1.0.10), part (i) can be written in the form

$$\left(\frac{1-\mu}{1+\mu}\right)^{10} \frac{\mu^6}{(1-\mu)^{12}} \left(\frac{1-3\mu+\mu^2}{\mu}\right)^5 = \frac{q}{\chi^{24}(q)}. \tag{2.5.9}$$

Apply Theorem 2.5.3 and (1.0.13), and proceed as in the proof of Theorem 2.3.4 to establish part (i). The proof of (ii) is similar. □

Corollary 2.5.5.

$$\text{(i)} \qquad \chi^4(q)\chi^4(q^5) = q\left(\frac{1+2\mu+\mu^2}{1-3\mu+\mu^2}\right)\frac{(1-\mu)^2}{\mu},$$

$$\text{(ii)} \qquad \frac{\chi^6(q)}{\chi^6(q^5)} = \frac{1}{q}\left(\frac{1+2\mu+\mu^2}{1-3\mu+\mu^2}\right)\frac{\mu}{(1-\mu)^2},$$

$$\text{(iii)} \qquad \frac{\chi^5(q)}{\chi(q^5)} = \frac{1+2\mu+\mu^2}{1-3\mu+\mu^2}.$$

Proof. The proof is analogous to the proof of Corollary 2.3.5 and follows from (1.0.10) and Theorem 2.5.4. □

We remark that M.S. Mahadeva Naika [66] has defined an analogue of μ for Ramanujan's cubic continued fraction, and has developed many analogous results for this parameter.

We record now analogues of Theorem 2.3.2, Theorem 2.3.4, and Corollary 2.3.5 for the parameter ν. Theorem 2.5.6(i) was discovered and proved first by Kang [7, Entry 1.8.1(ii), p. 33], [59, Theorem 4.1(ii)]. The remaining results in Theorem 2.5.6, Theorem 2.5.7, and Corollary 2.5.8 are new.

Theorem 2.5.6(i) follows from dividing part (vi) by part (v) in Theorem 2.5.1, dividing the numerator and denominator on the left-hand side of the resulting equation by $G^2(q)G(q^2)H(q^4)$, replacing q by $q^{1/2}$, and finally using (2.1.4) and (1.0.29). The proofs of the remaining parts of Theorem 2.5.6, Theorem 2.5.7, and Corollary 2.5.8 are analogous to the proofs of the corresponding results for the parameters μ and k, and so we omit the details.

Theorem 2.5.6. *We have*

$$\text{(i)} \qquad \frac{\psi(q)}{\sqrt{q}\psi(q^5)} = \frac{1+\nu}{1-\nu},$$

$$\text{(ii)} \qquad \frac{\chi^5(-q^5)}{q\chi(-q)} = \frac{4\nu}{(1-\nu)^2},$$

$$\text{(iii)} \qquad \frac{\phi^2(-q)}{\phi^2(-q^5)} = \frac{-\nu^2+3\nu-1}{\nu}.$$

Theorem 2.5.7. *If $q = e^{2\pi i\tau}$, where $\operatorname{Im}\tau > 0$, then*

$$\text{(i)} \qquad \frac{(1-\nu)^2}{4^6\nu}\left(\frac{(1+\nu)^2}{-\nu^2+3\nu-1}\right)^5 = q(-q;q)_\infty^{24} = \frac{\Delta(2\tau)}{\Delta(\tau)},$$

$$\text{(ii)} \qquad \frac{1}{4^6}\left(\frac{(1-\nu)^2}{\nu}\right)^5 \frac{(1+\nu)^2}{-\nu^2+3\nu-1} = q^5(-q^5;q^5)_\infty^{24} = \frac{\Delta(10\tau)}{\Delta(5\tau)}.$$

Corollary 2.5.8.

$$\text{(i)} \qquad \chi^4(-q)\chi^4(-q^5) = q\frac{4^2\nu(-\nu^2+3\nu-1)}{(1-\nu)^2(1+\nu)^2},$$

$$\text{(ii)} \qquad \frac{\chi^6(-q)}{\chi^6(-q^5)} = \frac{(-\nu^2+3\nu-1)(1-\nu)^2}{q\nu(1+\nu)^2},$$

$$\text{(iii)} \qquad \frac{\chi^5(-q)}{\chi(-q^5)} = \frac{4\left(-\nu^2+3\nu-1\right)}{(1+\nu)^2}.$$

Next, we apply our results to prove some remarkable modular identities relating $R(q)$, $R(-q)$, $R(q^2)$, and $R(q^4)$. Following Ramanujan, set

$$u = R(q), \qquad u' = -R(-q), \qquad v = R(q^2), \qquad \text{and} \qquad w = R(q^4). \tag{2.5.10}$$

Theorem 2.5.9. [7, Entry 1.5.1, pp. 24–25]

$$\text{(i)} \qquad uw = \frac{w-u^2v}{w+v^2},$$

$$\text{(ii)} \qquad uu'v^2 = \frac{uu'-v}{u'-u}.$$

The identities (i) and (ii) appear on page 205 of the Lost Notebook and were first proved in [21]. Before proving Theorem 2.5.9, we include a couple of remarks. Although (i) and (ii) appear different in form, they can be rewritten to evince a natural symmetry with each other. Indeed, by (2.5.10), (2.1.4), and (2.5.7),

$$u' = \frac{q^{1/5}H(-q)}{G(-q)} = \frac{q^{1/5}G(q)G(q^2)H(q^4)}{H(q)H(q^2)G(q^4)} = \frac{w}{uv}. \tag{2.5.11}$$

With the use of (2.5.11), it is easy to show that (i) is equivalent to

$$uu'v = \frac{u'-u}{uu'+v}. \tag{2.5.12}$$

Similarly, (ii) is equivalent to

$$\frac{vw}{u} = \frac{w-v^2}{w-u^2v}. \tag{2.5.13}$$

We further remark that by multiplying (i) with (2.5.13) and replacing q^2 by q, we recover Theorem 2.3.1(i).

Proof of Theorem 2.5.9(i). Take the quotient of Theorem 2.5.1(v) and (2.1.6) with q replaced by q^2 to find that

$$\frac{G^2(q)G(q^2)H(q^4)-H^2(q)H(q^2)G(q^4)}{G^2(q^2)H(q^4)+H^2(q^2)G(q^4)} = 2q\frac{\psi(q^{10})\psi(-q^5)}{f(-q^2)\psi(-q)}\frac{1}{2G(q^2)G^2(q^4)\frac{f_{20}^2}{f_{10}^2}}. \tag{2.5.14}$$

Use Lemma 1.0.1 and (2.3.6) to rewrite the right-hand side of (2.5.14) as

$$q\frac{G(q)H(q)H(q^4)}{G(q^2)G(q^4)}. \tag{2.5.15}$$

Substituting (2.5.15) into the right-hand side of (2.5.14) and multiplying the resulting equation by $G(q^2)/G^2(q)$, we find that

$$\frac{qH(q)H(q^4)}{G(q)G(q^4)} = \frac{G^2(q)G^2(q^2)H(q^4)-H^2(q)H(q^2)G(q^2)G(q^4)}{G^2(q)G^2(q^2)H(q^4)+G^2(q)H^2(q^2)G(q^4)}. \tag{2.5.16}$$

Divide the numerator and denominator on the right-hand side of (2.5.16) by

$$G(q^4)G^2(q^2)G^2(q)$$

and use (2.1.4) and (2.5.10) to arrive at Theorem 2.5.9(i). □

Proof of Theorem 2.5.9(ii). Use the equivalent form (2.5.13) and proceed as in the proof of (i), employing (2.1.5) with q replaced by q^2 in place of (2.1.6). □

Second Proof of Theorem 2.5.9(i), (ii). We establish (2.5.12) directly by employing (1.0.16). Using (2.1.4) in (2.5.12) and simplifying the result, we see that (2.5.12) is equivalent to

$$q\frac{H(q)H(-q)H(q^2)}{G(q)G(-q)G(q^2)} = G(q^2)\left(\frac{G(q)H(-q)-G(-q)H(q)}{G(q)G(-q)H(q^2)+H(q)H(-q)G(q^2)}\right). \tag{2.5.17}$$

Apply Theorem 2.5.1(iii) in the numerator of the right-hand side of (2.5.17), and apply (2.5.5), (2.1.2), and (2.1.3) in the denominator of the right-hand side of (2.5.17) to deduce that (2.5.12) is equivalent to

$$q\frac{H(q)H(-q)H(q^2)}{G(q)G(-q)G(q^2)} = \frac{2qG(q^2)\psi(q^{10})f(-q^2)}{f(q^4,q^6)f(-q^2,-q^8)+f(-q^4,-q^6)f(q^2,q^8)}. \tag{2.5.18}$$

Applying (1.0.16) with $a = q^4$, $b = q^6$, $c = -q^2$, and $d = -q^8$ in (2.5.18), we see that, in order to prove (2.5.12), it suffices to prove that

$$q\frac{H(q)H(-q)H(q^2)}{G(q)G(-q)G(q^2)} = \frac{2qG(q^2)\psi(q^{10})f(-q^2)}{2f(-q^6,-q^{14})f(-q^{12},-q^8)}. \tag{2.5.19}$$

Use (2.1.2), (2.1.3), (1.0.6), (1.0.12), and (1.0.9) to obtain the product expansion of each function above and to establish (2.5.19). We therefore establish (2.5.12), and hence also (i).

Part (ii) can similarly be established directly using the subtractive version of (1.0.16), namely (1.0.17). We omit the details. □

Given the beautiful analogies that we have demonstrated to exist among the results for k, μ, and ν, it is natural to ask whether there is a simple and elegant interrelationship among these three parameters. The answer is yes, and is recorded in part (i) of the next theorem. We note that this result is not recorded explicitly in this form in Ramanujan's work. We moreover provide two further, similar relations.

Theorem 2.5.10.

(i) $$\mu(q) + k(q) + \nu(q^2) - 1 = 0,$$

(ii) $$\frac{\mu(q) - k(q)}{1 - \nu(q^2)} = k(q^2),$$

(iii) $$\frac{\mu(q) - k(q)}{\mu(q) + k(q)} = k(q^2).$$

Proof. In Theorem 2.5.9(i), multiply by $w + v^2$ and then divide by w to arrive at

$$uw + uv^2 = 1 - \frac{u^2 v}{w}. \tag{2.5.20}$$

Using (2.5.10) and (1.0.27)–(1.0.29), and then rearranging, we arrive at (i). Similarly, part (ii) follows from the equivalent form of Theorem 2.5.9(ii) and (2.5.13). Part (iii) is immediate from parts (i) and (ii). Alternatively, replace q by q^2 in Theorem 2.3.1(i), multiply the numerator and denominator of the resulting equation by u, and use (2.5.10), (1.0.27), and (1.0.28) to obtain (iii). □

From Theorem 2.5.9 and Theorem 2.5.1(v), we derive new proofs of (2.1.5) and (2.1.6). We remark that although we employed (2.1.5) and (2.1.6) in our first proof of Theorem 2.5.9, our reasoning is not circular. Indeed, (2.1.5) and (2.1.6) were not utilized for our second proof of Theorem 2.5.9. See also [21, Theorem 2.1] for another proof of Theorem 2.5.9 independent of (2.1.5) and (2.1.6).

Theorem 2.5.11. *The modular identities* (2.1.5) *and* (2.1.6) *hold.*

Proof. Starting with the identity in Theorem 2.5.9(i), work the steps in the first proof of Theorem 2.5.9(i) in reverse order to deduce (2.5.14) from Theorem 2.5.9(i). Rearrange (2.5.14) to conclude that

$$\begin{aligned} G^2(q^2)H(q^4)+H^2(q^2)G(q^4) &= \left(G^2(q)G(q^2)H(q^4)-H^2(q)H(q^2)G(q^4)\right) \\ &\quad \times \left(\frac{f(-q^2)\psi(-q)}{2q\psi(q^{10})\psi(-q^5)}\right)\left(2G(q^2)G^2(q^4)\frac{f_{20}^2}{f_{10}^2}\right). \end{aligned} \tag{2.5.21}$$

Employing Theorem 2.5.1(v) in (2.5.21), we deduce that

$$G^2(q^2)H(q^4)+H^2(q^2)G(q^4)=2G(q^2)G^2(q^4)\frac{f_{20}^2}{f_{10}^2}. \tag{2.5.22}$$

Replacing q^2 by q in (2.5.22), we complete the proof of (2.1.6).

The identity (2.1.5) follows in a similar manner by beginning with (2.5.13), reversing the first proof of Theorem 2.5.9(ii), and applying Theorem 2.5.1(v). This completes the proof of Theorem 2.5.11. □

Next, we establish modular identities relating $R(q)$ and $R(q^4)$. Part (i) of Theorem 2.5.12 is new. Part (ii) appears on page 365 of Ramanujan's Lost Notebook, and was first proved in [16, Entry 5, pp. 18–19].

Theorem 2.5.12. *Let*

$$u=R(q) \qquad \textit{and} \qquad v=R(q^4). \tag{2.5.23}$$

Then

$$\text{(i)} \qquad \frac{\phi(q^{1/5})}{\phi(q^5)}=1+2\frac{u+v}{1-uv},$$

$$\text{(ii)} \qquad 5u^2v^2(uv-1)^2=(u^5+v^5)(uv-1)+u^5v^5+uv.$$

Proof of (i). We begin with a result from Ramanujan's notebooks [14, Entry 10(ii), p. 262], namely,

$$\phi(q^{1/5})-\phi(q^5)=2q^{1/5}f(q^3,q^7)+2q^{4/5}f(q,q^9). \tag{2.5.24}$$

Applying (2.1.2) and (2.1.3), we deduce from (2.5.3) and (2.5.4), respectively, that

$$f(q,q^9)=G(q)H(q^4)f(-q^2) \tag{2.5.25}$$

and

$$f(q^3,q^7)=G(q^4)H(q)f(-q^2). \tag{2.5.26}$$

Substituting (2.5.25) and (2.5.26) into (2.5.24), we find that

$$\phi(q^{1/5})-\phi(q^5)=2q^{1/5}G(q^4)H(q)f(-q^2)+2q^{4/5}G(q)H(q^4)f(-q^2). \tag{2.5.27}$$

Factor $2f(-q^2)G(q)G(q^4)$ out of each term on the right-hand side of (2.5.27), divide the resulting equation by $\phi(q^5)$, and finally apply (2.1.4) and (2.5.23) to deduce that

$$\frac{\phi(q^{1/5})}{\phi(q^5)}-1=\frac{2f(-q^2)G(q)G(q^4)}{\phi(q^5)}\left(u+v\right). \tag{2.5.28}$$

By Theorem 2.5.1(i), (2.1.4), and (2.5.23),

$$\begin{aligned}\frac{f(-q^2)G(q)G(q^4)}{\phi(q^5)} &= \frac{G(q)G(q^4)}{G(q)G(q^4)-qH(q)H(q^4)} \\ &= \frac{1}{1-\dfrac{qH(q)H(q^4)}{G(q)G(q^4)}} \\ &= \frac{1}{1-uv}. \qquad (2.5.29)\end{aligned}$$

Substituting (2.5.29) into (2.5.28) and rearranging the result, we deduce (i). □

Proof of (ii). The proof of (ii) is similar to the proof of (i). We begin with a result from Ramanujan's notebooks [14, Entry 10(vii), p. 262], namely,

$$32qf^5(q^3,q^7)+32q^4f^5(q,q^9) = \left(\frac{\phi^2(q)}{\phi(q^5)}-\phi(q^5)\right)\{\phi^4(q)-4\phi^2(q)\phi^2(q^5)+11\phi^4(q^5)\}. \qquad (2.5.30)$$

Apply (2.5.25) and (2.5.26) in (2.5.30), and then divide the resulting equation by $32G^5(q)G^5(q^4)f^5(-q^2)$. Use (2.1.4) and (2.5.23), and rearrange the right-hand side of the resulting equation to find that

$$u^5+v^5 = \frac{\phi^2(q)\phi^3(q^5)}{32f^5(-q^2)G^5(q)G^5(q^4)}\left(\frac{\phi^2(q)}{\phi^2(q^5)}-1\right)\left(\frac{\phi^2(q)}{\phi^2(q^5)}-4+11\frac{\phi^2(q^5)}{\phi^2(q)}\right). \qquad (2.5.31)$$

By Theorem 2.5.1(i), (ii), (2.1.4), and (1.0.28),

$$\begin{aligned}&\frac{\phi^2(q)\phi^3(q^5)}{f^5(-q^2)G^5(q)G^5(q^4)} \\ &\quad= \frac{1}{G^5(q)G^5(q^4)}\left(G(q)G(q^4)-qH(q)H(q^4)\right)^3\left(G(q)G(q^4)+qH(q)H(q^4)\right)^2 \\ &\quad= \left(1-\frac{qH(q)H(q^4)}{G(q)G(q^4)}\right)^3\left(1+\frac{qH(q)H(q^4)}{G(q)G(q^4)}\right)^2 \\ &\quad= (1-\mu)^3(1+\mu)^2. \qquad (2.5.32)\end{aligned}$$

Using (2.5.32) and Theorem 2.5.3(i) in (2.5.31) and simplifying, we conclude that

$$\begin{aligned}u^5+v^5 &= \frac{(1-\mu)^3(1+\mu)^2}{32}\left(\frac{(1+\mu)^2}{(1-\mu)^2}-1\right)\left(\frac{(1+\mu)^2}{(1-\mu)^2}-4+11\frac{(1-\mu)^2}{(1+\mu)^2}\right) \\ &= \frac{5\mu^2(\mu-1)^2-\mu^5-\mu}{\mu-1}. \qquad (2.5.33)\end{aligned}$$

Using the definition of μ, we easily see that (2.5.33) is equivalent to (ii). This completes the proof. □

Theorem 2.5.13. *Let*

$$u = R(q) \qquad \textit{and} \qquad u' = -R(-q). \qquad (2.5.34)$$

Then

$$\text{(i)} \qquad \frac{\psi(q^{2/5})}{q^{6/5}\psi(q^{10})} = 1 + 2\frac{1+uu'}{u'-u},$$

$$\text{(ii)} \qquad (u'-u)(1+u^5u'^5) = uu'(u'-u)^4 - u^2u'^2(u'-u)^2 + 2u^3u'^3.$$

Part (ii) appears on page 365 of Ramanujan's Lost Notebook, and was first proved in [16, Entry 2, p. 16]. Part (i) is new.

Proof of (i). We begin with

$$\psi(q^{2/5}) - q^{6/5}\psi(q^{10}) = f(q^4, q^6) + q^{2/5}f(q^2, q^8). \tag{2.5.35}$$

This result of Ramanujan is [14, Entry 10(i), p. 262] with q replaced by q^2. Employing (2.5.5) in (2.5.35), factoring out $G(q)G(-q)f(-q^2)$ from the right-hand side of the resulting equation, and then dividing the equation by $q^{6/5}\psi(q^{10})$, we find that

$$\frac{\psi(q^{2/5})}{q^{6/5}\psi(q^{10})} - 1 = \frac{G(q)G(-q)f(-q^2)}{q^{6/5}\psi(q^{10})}\left(1 + \frac{q^{2/5}H(q)H(-q)}{G(q)G(-q)}\right). \tag{2.5.36}$$

By Theorem 2.5.1(iii), (2.1.4), and (2.5.34),

$$\begin{aligned}\frac{G(q)G(-q)f(-q^2)}{q^{6/5}\psi(q^{10})} &= \frac{2G(q)G(-q)}{q^{1/5}\left(G(q)H(-q) - G(-q)H(q)\right)} \\ &= \frac{2}{\dfrac{q^{1/5}H(-q)}{G(-q)} - \dfrac{q^{1/5}H(q)}{G(q)}} \\ &= \frac{2}{u'-u}.\end{aligned} \tag{2.5.37}$$

Substituting (2.5.37) into (2.5.36) and employing (2.1.4) and (2.5.34), we find that

$$\frac{\psi(q^{2/5})}{q^{6/5}\psi(q^{10})} - 1 = 2\frac{1+uu'}{u'-u}. \tag{2.5.38}$$

A rearrangement of (2.5.38) yields (i). $\square$

Proof of (ii). We begin with

$$\begin{aligned}&f^5(q^4, q^6) + q^2 f^5(q^2, q^8) = \\ &\quad \left(\frac{\psi^2(q^2)}{\psi(q^{10})} - q^2\psi(q^{10})\right)\{\psi^4(q^2) - 4q^2\psi^2(q^2)\psi^2(q^{10}) + 11q^4\psi^4(q^{10})\}.\end{aligned} \tag{2.5.39}$$

This result of Ramanujan is [14, Entry 10(vi), p. 262] with q replaced by q^2. Use (2.5.5) and proceed as in the proof of Theorem 2.5.12(ii) to conclude from (2.5.39) that

$$\begin{aligned}&1 + u^5u'^5 = \\ &\quad \frac{q^4\psi^2(q^2)\psi^3(q^{10})}{f^5(-q^2)G^5(q)G^5(-q)}\left(\frac{\psi^2(q^2)}{q^2\psi^2(q^{10})} - 1\right)\left(\frac{\psi^2(q^2)}{q^2\psi^2(q^{10})} - 4 + 11\frac{q^2\psi^2(q^{10})}{\psi^2(q^2)}\right).\end{aligned} \tag{2.5.40}$$

For convenience, we now define

$$\nu_2 := \nu(q^2). \tag{2.5.41}$$

By (2.5.11),

$$u' = \frac{w}{u^2 v} u = \frac{u}{\nu_2}. \tag{2.5.42}$$

Hence,

$$uu' = \frac{u^2}{\nu_2} \qquad \text{and} \qquad u' \pm u = \frac{u}{\nu_2}\left(1 \pm \nu_2\right). \tag{2.5.43}$$

Furthermore, by Theorem 2.5.1(iv), (2.1.4), and (2.5.43),

$$\frac{\psi(q^2)}{f(-q^2)G(q)G(-q)} = \frac{1}{2q^{1/5}}\left(\frac{q^{1/5}H(-q)}{G(-q)} + \frac{q^{1/5}H(q)}{G(q)}\right) = \frac{u'+u}{2q^{1/5}} = \frac{u\left(1+\nu_2\right)}{2q^{1/5}\nu_2}. \tag{2.5.44}$$

Similarly, by Theorem 2.5.1(iii), (2.1.4), and (2.5.43),

$$\frac{\psi(q^{10})}{f(-q^2)G(q)G(-q)} = \frac{u\left(1-\nu_2\right)}{2q^{6/5}\nu_2}. \tag{2.5.45}$$

Consequently, employing (2.5.44), (2.5.45), and Theorem 2.5.6(i) in (2.5.40), and then using (2.5.43), we find that

$$\begin{aligned} 1 + u^5 u'^5 &= q^4 \frac{u^2(1+\nu_2)^2}{2^2 q^{2/5}\nu_2^2}\frac{u^3(1-\nu_2)^3}{2^3 q^{18/5}\nu_2^3}\left(\left(\frac{1+\nu_2}{1-\nu_2}\right)^2 - 1\right) \\ &\qquad \times \left(\left(\frac{1+\nu_2}{1-\nu_2}\right)^2 - 4 + 11\left(\frac{1-\nu_2}{1+\nu_2}\right)^2\right) \\ &= \frac{1}{(1-\nu_2)} \cdot \frac{u^5}{\nu_2^4}\left(1 - 5\nu_2 + 10\nu_2^2 - 5\nu_2^3 + \nu_2^4\right) \\ &= \frac{1}{(1-\nu_2)\dfrac{u}{\nu_2}} \cdot \frac{u^6}{\nu_2^5}\left(\left(1-\nu_2\right)^4 - \left(1-\nu_2\right)^2\nu_2 + 2\nu_2^2\right) \\ &= \frac{\dfrac{u^2}{\nu_2}\left(\left(1-\nu_2\right)\dfrac{u}{\nu_2}\right)^4 - \left(\dfrac{u^2}{\nu_2}\right)^2\left(\left(1-\nu_2\right)\dfrac{u}{\nu_2}\right)^2 + 2\left(\dfrac{u^2}{\nu_2}\right)^3}{\left(1-\nu_2\right)\dfrac{u}{\nu_2}} \\ &= \frac{uu'(u'-u)^4 - u^2u'^2(u'-u)^2 + 2u^3u'^3}{u'-u}. \end{aligned} \tag{2.5.46}$$

A rearrangement of (2.5.46) yields Theorem 2.5.13(ii). This completes the proof. ∎

Chapter 3

Cubic Modular Identities for the Rogers–Ramanujan and Göllnitz–Gordon Functions

3.1 Introduction

Our goal in this chapter is to study cubic modular relations for theta functions. In particular, we focus our attention on identities involving cubes of the Rogers–Ramanujan functions as well as certain analogues, namely the Göllnitz–Gordon functions, which we discuss next.

Recall the Rogers–Ramanujan functions and the Rogers–Ramanujan identities, (2.1.2) and (2.1.3). Two important analogues are the Göllnitz–Gordon functions, defined by

$$S(q) := \sum_{n=0}^{\infty} \frac{\left(-q;q^2\right)_n}{\left(q^2;q^2\right)_n} q^{n^2} = \frac{1}{\left(q;q^8\right)_\infty \left(q^4;q^8\right)_\infty \left(q^7;q^8\right)_\infty} \tag{3.1.1}$$

and

$$T(q) := \sum_{n=0}^{\infty} \frac{\left(-q;q^2\right)_n}{\left(q^2;q^2\right)_n} q^{n^2+2n} = \frac{1}{\left(q^3;q^8\right)_\infty \left(q^4;q^8\right)_\infty \left(q^5;q^8\right)_\infty}. \tag{3.1.2}$$

Identities (3.1.1) and (3.1.2) were first recorded by Ramanujan on page 41 of his Lost Notebook. They can also be found in L.J. Slater's list [89, Equations (36), (34)], but with q replaced by $-q$. Identities (3.1.1) and (3.1.2) are the analytic versions of the Göllnitz–Gordon identities [43], [45].

The Göllnitz–Gordon identities have played a seminal role in the subsequent development of the theory of partitions. They were first studied in this regard by H. Göllnitz [42, 43] and B. Gordon [44, 45]. A generalization by G.E. Andrews [1] led to a number of further discoveries culminating in [3].

In addition to the product-series identities (2.1.2), (2.1.3), (3.1.1), and (3.1.2), the Rogers–Ramanujan and Göllnitz–Gordon functions share further remarkable properties. For instance, S.–S. Huang [58] has derived an extensive list of elegant modular relations for the Göllnitz–Gordon functions analogous to Ramanujan's list of 40 identities for the Rogers–Ramanujan functions. S.–L. Chen and Huang [35] expanded on this list. Subsequently, N.D. Baruah, J. Bora, and N. Saikia [13] offered new proofs of many of the identities of Chen and Huang; their methods yielded further new relations as well.

For another shared property, recall that $q^{1/5}$ times the quotient of the Rogers–Ramanujan functions $H(q)$ and $G(q)$ gives a product representation of Ramanujan's famous continued fraction $R(q)$ (see (2.1.4)). Likewise, $q^{1/2}$ times the quotient of $T(q)$ and $S(q)$ yields the important Ramanujan–Göllnitz–Gordon continued fraction $K(q)$, given for $|q| < 1$ by

$$K(q) := \frac{q^{1/2}}{1+q} \, \underset{+}{} \, \frac{q^2}{1+q^3} \, \underset{+}{} \, \frac{q^4}{1+q^5} \, \underset{+}{} \, \frac{q^6}{1+q^7} \, \underset{+}{} \, \underset{\cdots}{}. \tag{3.1.3}$$

On page 229 of his second notebook [76, p. 229], Ramanujan recorded a product representation for $K(q)$, which we write in 3 different ways:

$$K(q) = q^{1/2}\frac{\left(q;q^8\right)_\infty \left(q^7;q^8\right)_\infty}{\left(q^3;q^8\right)_\infty \left(q^5;q^8\right)_\infty} = q^{1/2}\frac{f(-q,-q^7)}{f(-q^3,-q^5)} = q^{1/2}\frac{T(q)}{S(q)}. \tag{3.1.4}$$

Göllnitz [43] and Gordon [45] each independently rediscovered (3.1.4). Shortly thereafter, Andrews [2] proved (3.1.4) as a corollary of a more general result. Like the Rogers–Ramanujan continued fraction, $K(q)$ satisfies many beautiful modular relations. For example, in addition to (3.1.4), Ramanujan offered two other identities [76, p. 229] for $K(q)$, namely,

$$\frac{1}{K(q)} - K(q) = \frac{\phi(q^2)}{q^{1/2}\psi(q^4)} \tag{3.1.5}$$

and

$$\frac{1}{K(q)} + K(q) = \frac{\phi(q)}{q^{1/2}\psi(q^4)}, \tag{3.1.6}$$

where $\phi(q)$ and $\psi(q)$ are defined by (1.0.7) and (1.0.8), respectively. Proofs of (3.1.5) and (3.1.6) can be found in Berndt's book [14, p. 221]. The theory of the Ramanujan–Göllnitz–Gordon continued fraction has been further developed in recent years by various authors, including H.H. Chan and Huang [34]; K.R. Vasuki and B.R. Srivatsa Kumar [96]; and B. Cho, J.K. Koo, and Y.K. Park [36].

A fourth common property of the Rogers–Ramanujan and the Göllnitz–Gordon functions are their connections to the theory of partitions. In addition to the work of Andrews, Göllnitz, and Gordon already cited, Huang [58] has extracted a number of partition theorems from the modular relations that he derived for $S(q)$ and $T(q)$.

In this chapter, we develop further the theory of the Rogers–Ramanujan and Göllnitz–Gordon functions and their associated continued fractions. To this end, we present results as follows: In Section 3.2, we prove a number of theta function identities required for the sequel. Some of these identities appear to be new.

In Section 3.3, we prove two of our main results, namely two identities involving cubes of the Rogers–Ramanujan functions.

Next, in Section 3.4, we apply these two identities to derive new identities for the Rogers–Ramanujan continued fraction. As a consequence, we are able to offer a new proof of a modular relation of Ramanujan connecting $R(q)$ and $R(q^3)$. In particular, we prove that if $u = R(q)$ and $v = R(q^3)$, then [76, p. 321], [78, p. 365],

$$(v - u^3)(1 + uv^3) = 3u^2v^2. \tag{3.1.7}$$

Our methods yield simple, hitherto unknown identities for the two factors in parentheses on the left-hand side of (3.1.7).

Next, we demonstrate the generality of our methods by deriving new relations involving cubes of the Göllnitz–Gordon functions in Section 3.5. Further identities for $S(q)$ and $T(q)$ are developed in Section 3.6. The results of the preceding two sections are combined in Section 3.7 to prove, among other results, an analogue for $K(q)$ of (3.1.7). Lastly, in Section 3.8, we utilize the notion of colored partitions in order to extract partition-theoretic theorems from some of our results.

3.2 Auxiliary Theta Function Identities

The following theta function identity was stated by Ramanujan and first proved by Berndt [14, p. 349, Entry 2(ii)]. Subsequently, Baruah and Bora [10, Theorem 3.6] provided a different proof. Below, we offer a new proof that shows that (3.2.1) may be viewed as a consequence of the famous Quintuple Product Identity (1.0.19).

Lemma 3.2.1. *We have*

$$\psi(q) - 3q\psi(q^9) = \frac{\phi(-q)}{\chi(-q^3)}. \tag{3.2.1}$$

Proof. By [14, p. 49, Corollary(ii)], we know that

$$\psi(q) = f(q^3, q^6) + q\psi(q^9). \tag{3.2.2}$$

Applying (3.2.2), (1.0.8), and (1.0.3), we deduce that

$$\begin{aligned} \psi(q) - 3q\psi(q^9) &= f(q^3, q^6) - 2q\psi(q^9) \\ &= f(q^3, q^6) - qf(1, q^9). \end{aligned} \tag{3.2.3}$$

In the Quintuple Product Identity (1.0.19), replace q by $q^{3/2}$ and then set $B = q^{1/2}$ in order to find that

$$f(q^3, q^6) - qf(1, q^9) = f(-q^3)\frac{f(-q, -q^2)}{f(q, q^2)}. \tag{3.2.4}$$

Applying (1.0.9) six times, (1.0.6), and Lemma 1.0.1, we deduce that

$$f(-q^3) \cdot \frac{f(-q, -q^2)}{f(q, q^2)} = \left(q^3; q^3\right)_\infty \cdot \frac{f(-q)}{(-q; q^3)_\infty (-q^2; q^3)_\infty (q^3; q^3)_\infty}$$

$$
\begin{aligned}
&= f(-q)\cdot\frac{\left(-q^3;q^3\right)_\infty}{(-q;q)_\infty}\\
&= f(-q)\cdot\frac{(q;q)_\infty}{(q^2;q^2)_\infty}\cdot\frac{\left(q^6;q^6\right)_\infty}{(q^3;q^3)_\infty}\\
&= \frac{f^2(-q)}{f(-q^2)}\cdot\frac{f(-q^6)}{f(-q^3)}\\
&= \frac{\phi(-q)}{\chi(-q^3)}. \qquad (3.2.5)
\end{aligned}
$$

Employing (3.2.5) in (3.2.4) and combining the resulting equation with (3.2.3), we complete the proof. □

The first identity in the next lemma is [14, p. 379, Entry 10(ii)], and the second is [15, p. 188, Entry 36(ii)].

Lemma 3.2.2. *Let* $A = f(-q^4,-q^{11})$ *and* $B = qf(-q,-q^{14})$. *Then,*

(i) $$A - B = \frac{f(-q,-q^4)}{f(-q^2,-q^3)}f(-q^5),$$

(ii) $$A^3 - B^3 = \frac{f(-q^3,-q^{12})}{f(-q^6,-q^9)}f^3(-q^5).$$

In the following two theorems, we record new theta function identities that exhibit a high degree of symmetry. The theorems are applied in Sections 3.3 and 3.5, respectively.

Theorem 3.2.3. *Define*

$$
\begin{aligned}
&A := q^6 f(q^{12},q^{78}), && B := q^9 f(q^3,q^{87}), && C := q^4 f(q^{18},q^{72}),\\
&D := f(q^{33},q^{57}), && E := f(q^{42},q^{48}), && F := qf(q^{27},q^{63}).
\end{aligned}
$$

Furthermore, define

$$
\begin{aligned}
&\bar{A} := q^7 f(q^{9},q^{81}), && \bar{B} := q^8 f(q^6,q^{84}), && \bar{C} := q^3 f(q^{21},q^{69}),\\
&\bar{D} := f(q^{36},q^{54}), && \bar{E} := f(q^{39},q^{51}), && \bar{F} := q^2 f(q^{24},q^{66}).
\end{aligned}
$$

Then

(i) $AB - BC + CD - DE + EF - FA = -f(-q,-q^4)f(-q^{15},-q^{30})$,

(ii) $\bar{A}\bar{B} - \bar{B}\bar{C} + \bar{C}\bar{D} - \bar{D}\bar{E} + \bar{E}\bar{F} - \bar{F}\bar{A} = -f(-q^2,-q^3)f(-q^{15},-q^{30})$.

Proof. Applying (1.0.15) with $n = 3$, $a = -q$, and $b = -q^4$, we deduce that

$$
\begin{aligned}
f(-q,-q^4) &= f(-q^{18},-q^{27}) - qf(-q^{33},-q^{12}) + q^7 f(-q^{48},-q^{-3})\\
&= f(-q^{18},-q^{27}) - qf(-q^{33},-q^{12}) - q^4 f(-q^{3},-q^{42}), \qquad (3.2.6)
\end{aligned}
$$

where in the last line we applied (1.0.5). Next, we employ (1.0.18) successively with $a = -q^{18}$, $b = -q^{27}$, $c = -q^{15}$, $d = -q^{30}$; $a = -q^{12}$, $b = -q^{33}$, $c = -q^{15}$, $d = -q^{30}$; and $a = -q^3$, $b = -q^{42}$, $c = -q^{15}$, and $d = -q^{30}$, in order to deduce, respectively, that

$$f(-q^{18},-q^{27})f(-q^{15},-q^{30}) = f(q^{33},q^{57})f(q^{42},q^{48}) - q^{15}f(q^{12},q^{78})f(q^{3},q^{87}), \tag{3.2.7}$$

$$f(-q^{12},-q^{33})f(-q^{15},-q^{30}) = f(q^{27},q^{63})f(q^{42},q^{48}) - q^{12}f(q^{18},q^{72})f(q^{3},q^{87}), \tag{3.2.8}$$

and

$$f(-q^{3},-q^{42})f(-q^{15},-q^{30}) = f(q^{18},q^{72})f(q^{33},q^{57}) - q^{3}f(q^{27},q^{63})f(q^{12},q^{68}), \tag{3.2.9}$$

where we made one application of (1.0.5) in deducing the first line. Combining (3.2.7)–(3.2.9) with (3.2.6), we conclude that

$$\begin{aligned}
&f(-q,-q^4)f(-q^{15},-q^{30})\\
&\quad = f(-q^{18},-q^{27})f(-q^{15},-q^{30}) - qf(-q^{33},-q^{12})f(-q^{15},-q^{30})\\
&\qquad - q^4 f(-q^3,-q^{42})f(-q^{15},-q^{30})\\
&\quad = \Big(f(q^{33},q^{57})f(q^{42},q^{48}) - q^{15}f(q^{12},q^{78})f(q^{3},q^{87})\Big)\\
&\qquad - q\Big(f(q^{27},q^{63})f(q^{42},q^{48}) - q^{12}f(q^{18},q^{72})f(q^{3},q^{87})\Big)\\
&\qquad - q^4\Big(f(q^{18},q^{72})f(q^{33},q^{57}) - q^{3}f(q^{27},q^{63})f(q^{12},q^{78})\Big)\\
&\quad = DE - AB - FE + CB - CD + FA.
\end{aligned} \tag{3.2.10}$$

Rearranging (3.2.10), we deduce (i). The proof of (ii) is analogous; we omit the details. □

Remark. Z. Cao has kindly informed the author that the results in Theorem 3.2.3 may also be deduced as a corollary of a very general theta function identity that Cao has established [31, 32].

Theorem 3.2.4. *Set*

$$\begin{aligned}
&A := q^6 f(q^{30},q^{114}), && B := q^{14} f(q^{6},q^{138}), && C := q^{10} f(q^{18},q^{126}),\\
&D := q^2 f(q^{42},q^{102}), && E := f(q^{66},q^{78}), && F := f(q^{54},q^{90}).
\end{aligned}$$

Furthermore, set

$$\bar{A} := q^{12} f(q^{12},q^{132}), \qquad \bar{B} := q^{4} f(q^{36},q^{108}), \qquad \bar{C} = f(q^{60},q^{84}).$$

Then

(i) $$AB - BC + CD - DE + EF - FA = f(-q^{24})\psi(-q^2),$$

(ii) $$\bar{C} - \bar{A} = f(-q^{12}),$$

(iii) $$\bar{C} + \bar{A} - 2\bar{B} = \frac{\phi(-q^4)}{\chi(-q^{12})},$$

(iv) $$\bar{C}\bar{C} - \bar{A}\bar{A} + 2\bar{B}\bar{A} - 2\bar{B}\bar{C} = f(-q^{24})\phi(-q^4).$$

Proof of (i). By (1.0.9) and (1.0.8), the right-hand side of (i) is equal to

$$f(-q^{24},-q^{48})f(-q^2,-q^6). \tag{3.2.11}$$

Now apply (1.0.15) with $n=3$, $a=-q^2$, and $b=-q^6$, and proceed as in the proof of Theorem 3.2.3 to deduce (i). □

Proof of (ii). Apply (1.0.9) and (1.0.15) with $n=2$, $a=-q^{12}$, and $b=-q^{24}$. □

Proof of (iii). By (1.0.8) and (1.0.15) with $n=3$, $a=q$, and $b=q^3$, we find that

$$\psi(q)=f(q,q^3)=f(q^{15},q^{21})+q\psi(q^9)+q^3f(q^3,q^{33}). \tag{3.2.12}$$

We remark that Ramanujan explicitly recorded (3.2.12) in his notebooks [14, Corollary(ii), p. 49]. Replacing q by q^4 in (3.2.12), subtracting $3q^4\psi(q^{36})$ from both sides of the resulting equation, and then applying (1.0.8), we arrive at

$$\psi(q^4)-3q^4\psi(q^{36})=f(q^{60},q^{84})+q^{12}f(q^{12},q^{132})-2q^4\psi(q^{36})=\bar{C}+\bar{A}-2\bar{B}. \tag{3.2.13}$$

Replacing q by q^4 in Lemma 3.2.1 and substituting the result into (3.2.13), we complete the proof. □

Proof of (iv). The product of the left-hand sides of (ii) and (iii) yields the left-hand side of (iv). By Lemma 1.0.1, the product of the right-hand sides of (ii) and (iii) yields

$$f(-q^{12})\cdot\frac{\phi(-q^4)}{\chi(-q^{12})}=f(-q^{12})\cdot\frac{f(-q^{24})}{f(-q^{12})}\cdot\phi(-q^4)=f(-q^{24})\phi(-q^4),$$

which is the right-hand side of (iv). This completes the proof. □

Theorem 3.2.5. *Define*

$$A:=f(-q^7,-q^{17}),\qquad B:=q^2f(-q,-q^{23}),$$
$$C:=f(-q^{11},-q^{13}),\qquad D:=qf(-q^5,-q^{19}).$$

Then,

(i) $$A-B=\frac{f(-q^2,-q^6)}{f(-q^3,-q^5)}f(-q^8),$$

(ii) $$C+D=\frac{f(-q^2,-q^6)}{f(-q,-q^7)}f(-q^8),$$

(iii) $$A^3-B^3=\frac{f(-q^6,-q^{18})}{f(-q^9,-q^{15})}f^3(-q^8),$$

(iv) $$C^3+D^3=\frac{f(-q^6,-q^{18})}{f(-q^3,-q^{21})}f^3(-q^8),$$

(v) $$AD+BC=q\frac{f^2(-q^{24})}{f(-q^{12})}f(q),$$

(vi) $$AC+BD=\psi(q^3)f(-q^4).$$

Proofs of (i), (iii). Let ω denote an arbitrary third root of unity. In the Quintuple Product Identity (1.0.19), replace q by q^4 and set $B = -\omega q$. We consequently find that

$$\frac{f(-\omega^2 q^2, -\omega q^6)}{f(-\omega q^5, -\omega^2 q^3)} f(-q^8) = f(-q^7, -q^{17}) - \omega^2 q^2 f(-q, -q^{23}) = A - \omega^2 B. \tag{3.2.14}$$

Setting $\omega = 1$, we immediately deduce (i). Next, note that we obtain one identity from (3.2.14) for each distinct third root of unity ω. Multiplying together the three identities thus obtained, we find that

$$\prod_{\omega} \frac{f(-\omega^2 q^2, -\omega q^6)}{f(-\omega q^5, -\omega^2 q^3)} f(-q^8) = A^3 - B^3. \tag{3.2.15}$$

By applying the Jacobi Triple Product Identity (1.0.6) four times, we deduce that

$$\begin{aligned}\prod_{\omega} f(\omega a, \omega^2 b) &= f(a,b) f(\omega a, \omega^2 b) f(\omega^2 a, \omega b) \\ &= (-a; ab)_\infty (-b; ab)_\infty (-\omega a; ab)_\infty (-\omega^2 b; ab)_\infty (-\omega^2 a; ab)_\infty \\ &\quad \times (-\omega b; ab)_\infty (ab; ab)_\infty^3 \\ &= (-a^3; a^3 b^3)_\infty (-b^3; a^3 b^3)_\infty (ab; ab)_\infty^3 \\ &= \frac{f(a^3, b^3) f^3(-ab)}{f(-a^3 b^3)}. \end{aligned} \tag{3.2.16}$$

We note that (3.2.16) is also a special case of a general product formula due to S.H. Son [95, Theorem 3.1], [7, p. 14, Lemma 1.2.4]. Employing (3.2.16) with $a = -q^2,\ b = -q^6$, and with $a = -q^3,\ b = -q^5$, we readily deduce from (3.2.15) that

$$A^3 - B^3 = \frac{f(-q^6, -q^{18})}{f(-q^9, -q^{15})} f^3(-q^8). \tag{3.2.17}$$

This completes the proof of (iii). □

Proofs of (ii), (iv). In (1.0.19), replace q by q^4 and set $B = -\omega q^3$. Proceeding as in the proofs of (i) and (iii), we easily deduce (ii) and (iv). □

Proof of (v). By (1.0.6) and (1.0.9), it is not hard to show that

$$f(-q^5, -q^{19}) f(-q^7, -q^{17}) = f(-q^5, -q^7) \frac{f^2(-q^{24})}{f(-q^{12})} \tag{3.2.18}$$

and

$$f(-q, -q^{23}) f(-q^{11}, -q^{13}) = f(-q, -q^{11}) \frac{f^2(-q^{24})}{f(-q^{12})}. \tag{3.2.19}$$

Apply (3.2.18) and (3.2.19) to discover that

$$\begin{aligned} AD + BC &= q f(-q^7, -q^{17}) f(-q^5, -q^{19}) + q^2 f(-q, -q^{23}) f(-q^{11}, -q^{13}) \\ &= q \frac{f^2(-q^{24})}{f(-q^{12})} \left[f(-q^5, -q^7) + q f(-q, -q^{11}) \right]. \end{aligned} \tag{3.2.20}$$

In (1.0.15), set $a = q$, $b = -q^2$, and $n = 2$. Accordingly, we find that

$$f(q) = f(q, -q^2) = f(-q^5, -q^7) + qf(-q, -q^{11}). \tag{3.2.21}$$

Combining (3.2.21) with (3.2.20), we complete the proof. $\square$

Proof of (vi). Set $a = q^3$, $b = q^9$, $c = -q^4$, and $d = -q^8$ in (1.0.18). Applying also (1.0.8) and (1.0.9), we consequently find that

$$\begin{aligned} \psi(q^3)f(-q^4) &= f(q^3, q^9)f(-q^4, -q^8) \\ &= f(-q^7, -q^{17})f(-q^{11}, -q^{13}) + q^3 f(-q^5, -q^{19})f(-q, -q^{23}) \\ &= AC + BD. \end{aligned} \tag{3.2.22}$$

This completes the proof. $\square$

We require one further theorem for the Göllnitz–Gordon functions, namely

Lemma 3.2.6. *We have*

$$S(q) = \frac{f(-q^3, -q^5)}{\psi(-q)}, \qquad T(q) = \frac{f(-q, -q^7)}{\psi(-q)}, \tag{3.2.23}$$

and

$$S(q)T(q) = \frac{f^2(-q^8)f(-q^2)}{f^2(-q^4)f(-q)}. \tag{3.2.24}$$

The identities in Lemma 3.2.6 are easy consequences of (1.0.6), (1.0.8), (3.1.1), and (3.1.2).

3.3 Identities with Cubes of the Rogers–Ramanujan Functions

In this section, we prove identities involving cubes of the Rogers–Ramanujan functions. The identity in Theorem 3.3.1(i) is originally due to S. Robins [80], and has also been proved by W. Chu [37]. Robins employed the theory of modular forms while Chu's proof is a consequence of his work on q-difference equations. The remaining identities presented in this section are new.

Theorem 3.3.1. *The following identities hold:*

(i) $$G^3(q)H(q^3) - G(q^3)H^3(q) = 3q\frac{f^3(-q^{15})}{f(-q)f(-q^3)f(-q^5)},$$

(ii) $$G^3(q^3)G(q) + q^2H^3(q^3)H(q) = \frac{f^3(-q^5)}{f(-q)f(-q^3)f(-q^{15})}.$$

In Ramanujan's list of 40 identities for the Rogers–Ramanujan functions, he recorded several elegant relations for quotients of such identities. For example [19, p. 10, Equation (3.35)], Ramanujan found that

$$\frac{G(q^{17})H(q^2) - q^3H(q^{17})G(q^2)}{G(q^{34})G(q) + q^7H(q^{34})H(q)} = \frac{\chi(-q)}{\chi(-q^{17})}.$$

As an immediate corollary of Theorem 3.3.1, we have the following new identity involving a quotient of cubic identities for the Rogers–Ramanujan functions.

Corollary 3.3.2. *The following identity holds:*

$$\frac{G^3(q)H(q^3)-G(q^3)H^3(q)}{G^3(q^3)G(q)+q^2H^3(q^3)H(q)}=3q\frac{f^4(-q^{15})}{f^4(-q^5)}.$$

We now prove Theorem 3.3.1. We offer two proofs of (i), and one proof of (ii).

First Proof of (i). Applying (2.1.2)–(2.1.3), we rewrite (i) in the equivalent form

$$f^3(-q^2,-q^3)f(-q^3,-q^{12})-f^3(-q,-q^4)f(-q^6,-q^9)=3q\frac{f^3(-q^{15})f^2(-q)}{f(-q^5)}. \tag{3.3.1}$$

We prove (3.3.1). Let A and B be as in Lemma 3.2.2. Observe that, by Lemma 3.2.2,

$$\begin{aligned}\frac{1}{f^3(-q^5)}&\left(\frac{f(-q^3,-q^{12})}{f(-q^6,-q^9)}f^3(-q^5)-\frac{f^3(-q,-q^4)}{f^3(-q^2,-q^3)}f^3(-q^5)\right)\\ &=\frac{1}{f^3(-q^5)}\left(A^3-B^3-(A-B)^3\right)\\ &=\frac{1}{f^3(-q^5)}\left(3A^2B-3AB^2\right)\\ &=\frac{1}{f^3(-q^5)}3AB\,(A-B)\\ &=3q\frac{1}{f^3(-q^5)}f(-q^4,-q^{11})f(-q,-q^{14})\left(\frac{f(-q,-q^4)}{f(-q^2,-q^3)}f(-q^5)\right).\end{aligned} \tag{3.3.2}$$

Adding the fractions on the left-hand side of (3.3.2), we deduce that

$$\begin{aligned}&\frac{f^3(-q^2,-q^3)f(-q^3,-q^{12})-f^3(-q,-q^4)f(-q^6,-q^9)}{f(-q^6,-q^9)f^3(-q^2,-q^3)}\\ &\qquad=3q\frac{1}{f^2(-q^5)}f(-q^4,-q^{11})f(-q,-q^{14})\frac{f(-q,-q^4)}{f(-q^2,-q^3)}.\end{aligned} \tag{3.3.3}$$

Multiplying (3.3.3) by $f(-q^6,-q^9)f^3(-q^2,-q^3)$, applying the Jacobi Triple Product Identity (1.0.6) six times, and then simplifying with the help of (1.0.9), we conclude from (3.3.3) that

$$\begin{aligned}&f^3(-q^2,-q^3)f(-q^3,-q^{12})-f^3(-q,-q^4)f(-q^6,-q^9)\\ &=3q\frac{1}{f^2(-q^5)}f(-q^4,-q^{11})f(-q,-q^{14})f(-q,-q^4)f(-q^6,-q^9)f^2(-q^2,-q^3)\\ &=3q\frac{1}{f^2(-q^5)}(q^4;q^{15})_\infty(q^{11};q^{15})_\infty(q^{15};q^{15})_\infty(q;q^{15})_\infty(q^{14};q^{15})_\infty(q^{15};q^{15})_\infty\\ &\qquad\times(q;q^5)_\infty(q^4;q^5)_\infty(q^5;q^5)_\infty(q^6;q^{15})_\infty(q^9;q^{15})_\infty(q^{15};q^{15})_\infty\\ &\qquad\times(q^2;q^5)^2_\infty(q^3;q^5)^2_\infty(q^5;q^5)^2_\infty\\ &=3q\frac{1}{(q^5;q^5)^2_\infty}(q^{15};q^{15})^3_\infty(q;q)^2_\infty(q^5;q^5)_\infty\end{aligned}$$

$$= 3q\frac{f^3(-q^{15})f^2(-q)}{f(-q^5)}.$$

The last line establishes (3.3.1), and hence concludes the proof. □

Second Proof of (i). For this proof, we fix

$$\omega = e^{2\pi i/3}.$$

Now, applying (3.2.16) with $a = -q^2$, $b = -q^3$, and again with $a = -q$, $b = -q^4$, we deduce after rearranging that

$$f(-q^6, -q^9) = \frac{f(-q^{15})}{f^3(-q^5)} f(-q^2, -q^3) f(-\omega q^2, -\omega^2 q^3) f(-\omega^2 q^2, -\omega q^3) \tag{3.3.4}$$

and

$$f(-q^3, -q^{12}) = \frac{f(-q^{15})}{f^3(-q^5)} f(-q, -q^4) f(-\omega q, -\omega^2 q^4) f(-\omega^2 q, -\omega q^4). \tag{3.3.5}$$

As in the first proof of (i), we prove the equivalent formulation (3.3.1). Substituting (3.3.4) and (3.3.5) into (3.3.1), we see that, in order to prove (3.3.1), it is sufficient to prove that

$$\begin{aligned}
&\frac{f(-q^{15})}{f^3(-q^5)} \Big(f^3(-q^2, -q^3) f(-q, -q^4) f(-\omega q, -\omega^2 q^4) f(-\omega^2 q, -\omega q^4) \\
&- f^3(-q, -q^4) f(-q^2, -q^3) f(-\omega q^2, -\omega^2 q^3) f(-\omega^2 q^2, -\omega q^3) \Big) \\
&= 3q \frac{f^3(-q^{15}) f^2(-q)}{f(-q^5)}.
\end{aligned} \tag{3.3.6}$$

By (1.0.6), it is easy to verify that

$$f(-q, -q^4) f(-q^2, -q^3) = f(-q) f(-q^5). \tag{3.3.7}$$

Applying (3.3.7) twice in (3.3.6) and simplifying, we deduce that (3.3.1) is equivalent to

$$\begin{aligned}
&f^2(-q^2, -q^3) f(-\omega q, -\omega^2 q^4) f(-\omega^2 q, -\omega q^4) \\
&\quad - f^2(-q, -q^4) f(-\omega q^2, -\omega^2 q^3) f(-\omega^2 q^2, -\omega q^3) = 3q f^2(-q^{15}) f(-q^5) f(-q).
\end{aligned} \tag{3.3.8}$$

Applying Lemma 1.0.2, first with $a = -\omega q$, $b = -\omega^2 q^4$, $c = -q^2$, $d = -q^3$; secondly with $a = -\omega^2 q$, $b = -\omega q^4$, $c = -q^2$, $d = -q^3$; thirdly with $a = -q$, $b = -q^4$, $c = -\omega q^2$, $d = -\omega^2 q^3$; and fourthly with $a = -q$, $b = -q^4$, $c = -\omega^2 q^2$, $d = -\omega q^3$, we deduce, respectively, that

$$\begin{aligned}
f(-\omega q, -\omega^2 q^4) f(-q^2, -q^3) =& f(\omega q^3, \omega^2 q^7) f(\omega q^4, \omega^2 q^6) \\
&- \omega q f(\omega^2 q^2, \omega q^8) f(\omega^2 q, \omega q^9),
\end{aligned} \tag{3.3.9}$$

$$\begin{aligned}
f(-\omega^2 q, -\omega q^4) f(-q^2, -q^3) =& f(\omega^2 q^3, \omega q^7) f(\omega^2 q^4, \omega q^6) \\
&- \omega^2 q f(\omega q^2, \omega^2 q^8) f(\omega q, \omega^2 q^9),
\end{aligned} \tag{3.3.10}$$

$$\begin{aligned}
f(-q, -q^4) f(-\omega q^2, -\omega^2 q^3) =& f(\omega q^3, \omega^2 q^7) f(\omega^2 q^4, \omega q^6) \\
&- q f(\omega^2 q^2, \omega q^8) f(\omega q, \omega^2 q^9),
\end{aligned} \tag{3.3.11}$$

and

$$f(-q,-q^4)f(-\omega^2q^2,-\omega q^3) = f(\omega^2q^3,\omega q^7)f(\omega q^4,\omega^2q^6) \\ - qf(\omega q^2,\omega^2q^8)f(\omega^2q,\omega q^9). \tag{3.3.12}$$

Substituting (3.3.9)–(3.3.12) into (3.3.8), we see that (3.3.1) is equivalent to

$$\begin{aligned}
&\Big(f(\omega q^3,\omega^2q^7)f(\omega q^4,\omega^2q^6)-\omega qf(\omega^2q^2,\omega q^8)f(\omega^2q,\omega q^9)\Big)\\
&\times\Big(f(\omega^2q^3,\omega q^7)f(\omega^2q^4,\omega q^6)-\omega^2qf(\omega q^2,\omega^2q^8)f(\omega q,\omega^2q^9)\Big)\\
&-\Big(f(\omega q^3,\omega^2q^7)f(\omega^2q^4,\omega q^6)-qf(\omega^2q^2,\omega q^8)f(\omega q,\omega^2q^9)\Big)\\
&\times\Big(f(\omega^2q^3,\omega q^7)f(\omega q^4,\omega^2q^6)-qf(\omega q^2,\omega^2q^8)f(\omega^2q,\omega q^9)\Big)\\
&\quad=3qf^2(-q^{15})f(-q^5)f(-q).
\end{aligned} \tag{3.3.13}$$

Expanding out the left-hand side of (3.3.13) and performing the obvious simplifications, we deduce that, in order to prove (3.3.1), it suffices to prove that

$$\begin{aligned}
&f(\omega q^3,\omega^2q^7)f(\omega^2q^4,\omega q^6)f(\omega q^2,\omega^2q^8)f(\omega^2q,\omega q^9)\\
&+f(\omega^2q^3,\omega q^7)f(\omega q^4,\omega^2q^6)f(\omega^2q^2,\omega q^8)f(\omega q,\omega^2q^9)\\
&-\omega^2f(\omega q^3,\omega^2q^7)f(\omega q^4,\omega^2q^6)f(\omega q^2,\omega^2q^8)f(\omega q,\omega^2q^9)\\
&-\omega f(\omega^2q^3,\omega q^7)f(\omega^2q^4,\omega q^6)f(\omega^2q^2,\omega q^8)f(\omega^2q,\omega q^9)\\
&\quad=3f^2(-q^{15})f(-q^5)f(-q).
\end{aligned} \tag{3.3.14}$$

Next, we employ (1.0.15) with $n=3$ along with one application of (1.0.5) to deduce that

$$f(a,b)=f(a^3b^6,a^6b^3)+af(b^3,a^9b^6)+bf(a^3,a^6b^9). \tag{3.3.15}$$

Choosing $a=\omega q^3$ and $b=\omega^2q^7$ in (3.3.15), we find that

$$f(\omega q^3,\omega^2q^7)=f(q^{51},q^{39})+\omega q^3f(q^{21},q^{69})+\omega^2q^7f(q^9,q^{81}). \tag{3.3.16}$$

Replacing ω by ω^2 in (3.3.16), we find that

$$f(\omega^2q^3,\omega q^7)=f(q^{51},q^{39})+\omega^2q^3f(q^{21},q^{69})+\omega q^7f(q^9,q^{81}). \tag{3.3.17}$$

Similarly, we deduce the relations

$$f(\omega q^4,\omega^2q^6)=f(q^{48},q^{42})+\omega q^4f(q^{18},q^{72})+\omega^2q^6f(q^{12},q^{78}), \tag{3.3.18}$$
$$f(\omega^2q^4,\omega q^6)=f(q^{48},q^{42})+\omega^2q^4f(q^{18},q^{72})+\omega q^6f(q^{12},q^{78}), \tag{3.3.19}$$
$$f(\omega q^2,\omega^2q^8)=f(q^{54},q^{36})+\omega q^2f(q^{24},q^{66})+\omega^2q^8f(q^6,q^{84}), \tag{3.3.20}$$
$$f(\omega^2q^2,\omega q^8)=f(q^{54},q^{36})+\omega^2q^2f(q^{24},q^{66})+\omega q^8f(q^6,q^{84}), \tag{3.3.21}$$
$$f(\omega q,\omega^2q^9)=f(q^{57},q^{33})+\omega qf(q^{27},q^{63})+\omega^2q^9f(q^3,q^{87}), \tag{3.3.22}$$

and

$$f(\omega^2q,\omega q^9)=f(q^{57},q^{33})+\omega^2qf(q^{27},q^{63})+\omega q^9f(q^3,q^{87}). \tag{3.3.23}$$

Substituting (3.3.16)–(3.3.23) into the left-hand side of (3.3.14), expanding the resulting expression, simplifying, and then factoring with the help of MAPLE, we deduce that (3.3.1) is equivalent to

$$\begin{aligned}
&3\Big[q^{15}f(q^{12},q^{78})f(q^{3},q^{87})-q^{13}f(q^{18},q^{72})f(q^{3},q^{87})+qf(q^{42},q^{48})f(q^{27},q^{63})\\
&-f(q^{42},q^{48})f(q^{33},q^{57})-q^{7}f(q^{12},q^{78})f(q^{27},q^{63})+q^{4}f(q^{18},q^{72})f(q^{33},q^{57})\Big]\\
&\times\Big[q^{15}f(q^{9},q^{81})f(q^{6},q^{84})-q^{11}f(q^{21},q^{69})f(q^{6},q^{84})-q^{9}f(q^{9},q^{81})f(q^{24},q^{26})\\
&+q^{3}f(q^{36},q^{54})f(q^{21},q^{69})+q^{2}f(q^{39},q^{51})f(q^{24},q^{66})-f(q^{36},q^{54})f(q^{39},q^{51})\Big]\\
&=3f^{2}(-q^{15})f(-q^{5})f(-q). \qquad (3.3.24)
\end{aligned}$$

Apply Theorem 3.2.3 to the two bracketed expressions above. Accordingly, we deduce that (3.3.1) is equivalent to

$$\begin{aligned}
&3[-f(-q,-q^{4})f(-q^{15},-q^{30})]\times[-f(-q^{2},-q^{3})f(-q^{15},-q^{30})]\\
&\quad=3f^{2}(-q^{15})f(-q^{5})f(-q). \qquad (3.3.25)
\end{aligned}$$

Applying (1.0.9) and (3.3.7), we readily deduce the truth of (3.3.25). This then proves (3.3.1), and hence completes our second proof of (i).

□

Proof of (ii). Applying (2.1.2)–(2.1.3), we rewrite (ii) in the equivalent form

$$f^{3}(-q^{6},-q^{9})f(-q^{2},-q^{3})+q^{2}f^{3}(-q^{3},-q^{12})f(-q,-q^{4})=\frac{f^{3}(-q^{5})f^{2}(-q^{3})}{f(-q^{15})}. \qquad (3.3.26)$$

We prove (3.3.26). By (1.0.6), we verify that

$$f(-q^{2},-q^{3})=f(-q^{2},-q^{13})f(-q^{3},-q^{12})f(-q^{7},-q^{8})\frac{f(-q^{5})}{f^{3}(-q^{15})} \qquad (3.3.27)$$

and

$$f(-q,-q^{4})=f(-q,-q^{14})f(-q^{6},-q^{9})f(-q^{4},-q^{11})\frac{f(-q^{5})}{f^{3}(-q^{15})}. \qquad (3.3.28)$$

Identities (3.3.27)–(3.3.28) are also recorded explicitly in Ramanujan's notebooks [14, p. 222, Entry 2(i)]. Employing these, we find that (3.3.26) is equivalent to

$$\begin{aligned}
&f^{3}(-q^{6},-q^{9})f(-q^{2},-q^{13})f(-q^{3},-q^{12})f(-q^{7},-q^{8})\frac{f(-q^{5})}{f^{3}(-q^{15})}\\
&+q^{2}f^{3}(-q^{3},-q^{12})f(-q,-q^{14})f(-q^{6},-q^{9})f(-q^{4},-q^{11})\frac{f(-q^{5})}{f^{3}(-q^{15})}\\
&=\frac{f^{3}(-q^{5})f^{2}(-q^{3})}{f(-q^{15})}. \qquad (3.3.29)
\end{aligned}$$

Replacing q by q^3 in (3.3.7), we see that

$$f(-q^{3},-q^{12})f(-q^{6},-q^{9})=f(-q^{3})f(-q^{15}). \qquad (3.3.30)$$

Applying (3.3.30) twice in (3.3.29) and simplifying, we deduce that (3.3.26) is equivalent to

$$\begin{aligned}&f^2(-q^6,-q^9)f(-q^2,-q^{13})f(-q^7,-q^8)\\&+q^2f^2(-q^3,-q^{12})f(-q,-q^{14})f(-q^4,-q^{11})=f(-q^{15})f^2(-q^5)f(-q^3).\end{aligned} \tag{3.3.31}$$

Next, we apply Lemma 1.0.2 four times, first with $a=-q^2$, $b=-q^{13}$, $c=-q^6$, $d=-q^9$; secondly with $a=-q^7$, $b=-q^8$, $c=-q^6$, $d=-q^9$; thirdly with $a=-q$, $b=-q^{14}$, $c=-q^3$, $d=-q^{12}$; and fourthly with $a=-q^4$, $b=-q^{11}$, $c=-q^3$, $d=-q^{12}$, along with (1.0.5) in the second and fourth cases. This yields the identities

$$f(-q^2,-q^{13})f(-q^6,-q^9)=f(q^8,q^{22})f(q^{11},q^{19})-q^2f(q^7,q^{23})f(q^4,q^{26}), \tag{3.3.32}$$

$$f(-q^7,-q^8)f(-q^6,-q^9)=f(q^{13},q^{17})f(q^{14},q^{16})-q^6f(q^2,q^{28})f(q,q^{29}), \tag{3.3.33}$$

$$f(-q,-q^{14})f(-q^3,-q^{12})=f(q^4,q^{26})f(q^{13},q^{17})-qf(q^{11},q^{19})f(q^2,q^{28}), \tag{3.3.34}$$

$$f(-q^4,-q^{11})f(-q^3,-q^{12})=f(q^7,q^{23})f(q^{14},q^{16})-q^3f(q^8,q^{22})f(q,q^{29}). \tag{3.3.35}$$

Substituting (3.3.32)–(3.3.35) into (3.3.31), we see that (3.3.26) is equivalent to

$$\begin{aligned}&\Big[f(q^8,q^{22})f(q^{11},q^{19})-q^2f(q^7,q^{23})f(q^4,q^{26})\Big]\\&\times\Big[f(q^{13},q^{17})f(q^{14},q^{16})-q^6f(q^2,q^{28})f(q,q^{29})\Big]\\&+q^2\Big[f(q^4,q^{26})f(q^{13},q^{17})-qf(q^{11},q^{19})f(q^2,q^{28})\Big]\\&\times\Big[f(q^7,q^{23})f(q^{14},q^{16})-q^3f(q^8,q^{22})f(q,q^{29})\Big]=f(-q^{15})f^2(-q^5)f(-q^3).\end{aligned} \tag{3.3.36}$$

Expanding out the left-hand side of (3.3.36), performing the obvious cancellations, and then factoring the resulting expression, we find that (3.3.26) is equivalent to

$$\begin{aligned}&\Big[f(q^{11},q^{19})f(q^{14},q^{16})-q^5f(q^4,q^{26})f(q,q^{29})\Big]\\&\times\Big[f(q^8,q^{22})f(q^{13},q^{17})-q^3f(q^2,q^{28})f(q^7,q^{23})\Big]=f(-q^{15})f^2(-q^5)f(-q^3).\end{aligned} \tag{3.3.37}$$

Next, we apply Lemma 1.0.2 twice more, first with $a=-q^5$, $b=-q^{10}$, $c=-q^6$, $d=-q^9$; and secondly with $a=-q^3$, $b=-q^{12}$, $c=-q^5$, $d=-q^{10}$. Accordingly, we deduce that

$$f(-q^6,-q^9)f(-q^5,-q^{10})=f(q^{11},q^{19})f(q^{14},q^{16})-q^5f(q^4,q^{26})f(q,q^{29}), \tag{3.3.38}$$

$$f(-q^3,-q^{12})f(-q^5,-q^{10})=f(q^8,q^{22})f(q^{13},q^{17})-q^3f(q^7,q^{23})f(q^2,q^{28}). \tag{3.3.39}$$

Substituting (3.3.38)–(3.3.39) into (3.3.37), we see that, in order to prove (3.3.26), it suffices to prove that

$$\left[f(-q^6,-q^9)f(-q^5,-q^{10})\right]\times\left[f(-q^3,-q^{12})f(-q^5,-q^{10})\right]$$
$$= f(-q^{15})f^2(-q^5)f(-q^3). \tag{3.3.40}$$

Employing (1.0.9) and (3.3.30), we readily deduce the truth of (3.3.40). This completes the proof. □

3.4 Applications to the Rogers–Ramanujan Continued Fraction

We begin by offering new identities for the Rogers–Ramanujan continued fraction.

Theorem 3.4.1. *Let* $u = R(q)$ *and* $v = R(q^3)$. *Then*

(i) $$v - u^3 = 3q^{8/5}\frac{f^3(-q^{15})}{f(-q)f(-q^3)f(-q^5)G^3(q)G(q^3)},$$

(ii) $$1 + uv^3 = \frac{f^3(-q^5)}{f(-q)f(-q^3)f(-q^{15})G(q)G^3(q^3)},$$

(iii) $$\frac{v-u^3}{1+uv^3} = 3q^2\frac{f^5(-q^{15})f(-q)}{f^5(-q^5)f(-q^3)}\cdot\frac{u}{v},$$

(iv) $$\frac{v}{u^3}+\frac{u^3}{v} = 9q^2\frac{f^5(-q^{15})f(-q)}{f^5(-q^5)f(-q^3)}+2 = 3\frac{v-u^3}{1+uv^3}\cdot\frac{v}{u}+2,$$

(v) $$\frac{1}{uv^3}+uv^3 = \frac{f^5(-q^5)f(-q^3)}{q^2f^5(-q^{15})f(-q)}-2 = 3\frac{1+uv^3}{v-u^3}\cdot\frac{u}{v}-2.$$

Proof of (i),(ii). Multiply the identity in Theorem 3.3.1(i) by $q^{3/5}G^{-3}(q)G^{-1}(q^3)$ and apply (2.1.4). Accordingly, we deduce (i). Similarly, multiplying the identity in Theorem 3.3.1(ii) by $G^{-1}(q)G^{-3}(q^3)$ and applying (2.1.4), we deduce (ii). □

Proof of (iii). Divide the identities in (i) and (ii). The left-hand side of (iii) is immediate. Observe that, by (2.3.6), and (2.1.4), the right-hand side of the resulting identity is equal to

$$\begin{aligned}3q^{8/5}\frac{f^4(-q^{15})}{f^4(-q^5)}\frac{G^2(q^3)}{G^2(q)} &= 3q^{8/5}\frac{f^4(-q^{15})}{f^4(-q^5)}\cdot\frac{f(-q)}{f(-q^5)}\frac{f(-q^{15})}{f(-q^3)}\frac{G(q)H(q)}{G(q^3)H(q^3)}\cdot\frac{G^2(q^3)}{G^2(q)}\\ &= 3q^2\frac{f^5(-q^{15})f(-q)}{f^5(-q^5)f(-q^3)}\cdot\frac{G(q^3)}{q^{3/5}H(q^3)}\cdot\frac{q^{1/5}H(q)}{G(q)}\\ &= 3q^2\frac{f^5(-q^{15})f(-q)}{f^5(-q^5)f(-q^3)}\cdot\frac{u}{v}.\end{aligned}$$

This completes the proof. □

Proof of (iv),(v). Square the identity in (i), divide the resulting equation by u^3v, and rearrange the result to find that

$$\frac{v}{u^3}+\frac{u^3}{v} = 9q^{16/5}\frac{f^6(-q^{15})}{f^2(-q)f^2(-q^3)f^2(-q^5)G^6(q)G^2(q^3)u^3v}+2. \tag{3.4.1}$$

With the help of (2.1.4) and (2.3.6), we deduce that

$$\begin{aligned}
&9q^{16/5}\frac{f^6(-q^{15})}{f^2(-q)f^2(-q^3)f^2(-q^5)G^6(q)G^2(q^3)u^3v}\\
&=9q^{16/5}\frac{f^6(-q^{15})}{f^2(-q)f^2(-q^3)f^2(-q^5)G^6(q)G^2(q^3)}\cdot\frac{G^3(q)G(q^3)}{q^{6/5}H^3(q)H(q^3)}\\
&=9q^2\frac{f^6(-q^{15})}{f^2(-q)f^2(-q^3)f^2(-q^5)}\cdot\frac{1}{G^3(q)H^3(q)G(q^3)H(q^3)}\\
&=9q^2\frac{f^6(-q^{15})}{f^2(-q)f^2(-q^3)f^2(-q^5)}\cdot\frac{f^3(-q)f(-q^3)}{f^3(-q^5)f(-q^{15})}\\
&=9q^2\frac{f^5(-q^{15})f(-q)}{f^5(-q^5)f(-q^3)}. \qquad (3.4.2)
\end{aligned}$$

Combining (3.4.1) with (3.4.2), we deduce the first equality in (iv). To prove the second equality, observe that, by (iii),

$$3q^2\frac{f^5(-q^{15})f(-q)}{f^5(-q^5)f(-q^3)}=\frac{v-u^3}{1+uv^3}\cdot\frac{v}{u}. \qquad (3.4.3)$$

Substituting (3.4.3) into the first equality of (iv), we deduce the second equality.

Analogously, we deduce (v) from (ii). This completes the proof. □

In his notebooks, Ramanujan recorded the following exquisite modular equation of degree three [76, p. 321], [16, p. 17]; see also Ramanujan's Lost Notebook [78, p. 365], [7, p. 92].

Theorem 3.4.2. *Let*

$$u:=\frac{q^{1/5}}{1}\,+\,\frac{q}{1}\,+\,\frac{q^2}{1}\,+\,\frac{q^3}{1}\,+\cdots$$

and

$$v:=\frac{q^{3/5}}{1}\,+\,\frac{q^3}{1}\,+\,\frac{q^6}{1}\,+\,\frac{q^9}{1}\,+\cdots.$$

Then

$$(v-u^3)(1+uv^3)=3u^2v^2.$$

Only two proofs of Theorem 3.4.2 are known in the literature. The first is due to Rogers [82, p. 392, Equation (6.2)], who uses the classical theory of theta functions. The second is due to Yi [102], who utilizes eta function identities. We offer a new proof, based on Theorem 3.4.1.

Proof. Observe that, by Theorem 3.4.1(i),(ii), it suffices to prove that

$$\frac{3q^{8/5}f^3(-q^{15})}{f(-q)f(-q^3)f(-q^5)G^3(q)G(q^3)}\cdot\frac{f^3(-q^5)}{f(-q)f(-q^3)f(-q^{15})G(q)G^3(q^3)}=3u^2v^2. \qquad (3.4.4)$$

With the use of (2.3.6) and (2.1.4), we readily deduce that the left-hand side of (3.4.4) is equal to

$$\begin{aligned}
&3q^{8/5}\cdot\frac{f^2(-q^5)}{f^2(-q)}\cdot\frac{f^2(-q^{15})}{f^2(-q^3)}\cdot\frac{1}{G^4(q)G^4(q^3)}\\
&\quad=3q^{8/5}\cdot G^2(q)H^2(q)\cdot G^2(q^3)H^2(q^3)\cdot\frac{1}{G^4(q)G^4(q^3)}\\
&\quad=3\cdot\frac{q^{2/5}H^2(q)}{G^2(q)}\cdot\frac{q^{6/5}H^2(q^3)}{G^2(q^3)}\\
&\quad=3u^2v^2. \qquad (3.4.5)
\end{aligned}$$

This proves (3.4.4), and thus completes the proof. □

We remark that our proof is the first to give identities for the factors appearing on the left-hand side of Theorem 3.4.2.

Next, we discuss a couple further results of Ramanujan and corollaries of our work. The following lemma is stated on page 205 of Ramanujan's Lost Notebook, and has been proved by J. Sohn [91].

Lemma 3.4.3. [7, p. 45, Entry 1.10.2], [91] *Let* $\omega=\exp(2\pi i/3)$, $u=R(q)$, *and* $v=R(q^3)$. *If*

$$R:=\frac{f^2(-q^3)}{qf^2(-q^{15})}=\left(\frac{1}{v^5}-11-v^5\right)^{1/3},$$

then

$$4u=-\frac{1}{v^3}-\sqrt{\frac{1}{v^6}-\frac{8+4R}{v}}+\sqrt{\frac{1}{v^6}-\frac{8+4R\omega}{v}}+\sqrt{\frac{1}{v^6}-\frac{8+4R\omega^2}{v}}. \quad (3.4.6)$$

If

$$R:=\frac{f^2(-q)}{q^{1/3}f^2(-q^5)}=\left(\frac{1}{u^5}-11-u^5\right)^{1/3},$$

then

$$4v=u^3-\sqrt{u^6+u(8+4R)}+\sqrt{u^6+(8+4R\omega)}+\sqrt{u^6+u(8+4R\omega^2)}. \quad (3.4.7)$$

From Lemma 3.4.3 and Theorem 3.4.1, we obtain the following corollary.

Corollary 3.4.4. *Let* $\omega=\exp(2\pi i/3)$, $u=R(q)$, *and* $v=R(q^3)$. *If*

$$R:=\frac{f^2(-q^3)}{qf^2(-q^{15})}=\left(\frac{1}{v^5}-11-v^5\right)^{1/3},$$

then

$$\begin{aligned}
&\frac{4f^3(-q^5)}{f(-q)f(-q^3)f(-q^{15})}=G(q)G^3(q^3)\\
&\times\left(3-v^3\sqrt{\frac{1}{v^6}-\frac{8+4R}{v}}+v^3\sqrt{\frac{1}{v^6}-\frac{8+4R\omega}{v}}+v^3\sqrt{\frac{1}{v^6}-\frac{8+4R\omega^2}{v}}\right).
\end{aligned} \quad (3.4.8)$$

If

$$R := \frac{f^2(-q)}{q^{1/3}f^2(-q^5)} = \left(\frac{1}{u^5} - 11 - u^5\right)^{1/3},$$

then

$$\frac{12q^{8/5}f^3(-q^{15})}{f(-q)f(-q^3)f(-q^5)} = G^3(q)G(q^3)$$
$$\times\left(-3u^3 - \sqrt{u^6+u(8+4R)} + \sqrt{u^6+(8+4R\omega)} + \sqrt{u^6+u(8+4R\omega^2)}\right). \tag{3.4.9}$$

Proof. Multiplying (3.4.6) by v^3, adding 4 to both sides of the resulting equation, and then employing Theorem 3.4.1(ii), we readily deduce (3.4.8). Similarly, (3.4.9) follows from (3.4.7) and Theorem 3.4.1(i). □

In his notebooks, Ramanujan recorded the following entry connected with u and v.

Lemma 3.4.5. [16, Entry 4, p. 17] *Let u and v be defined as above. Let*

$$m := q^{2/5}\frac{f(-q^4,-q^{11})}{f(-q^7,-q^8)}v$$

and

$$n := q^{2/5}\frac{f(-q,-q^{14})}{f(-q^2,-q^{13})}v.$$

Then

$$m - n = mn = \frac{m^2}{1+m} = \frac{n^2}{1-n} = uv^3.$$

See [16, pp. 17-18] for a proof. From Lemma 3.4.5 and Theorem 3.4.1(ii), we easily deduce the following corollary.

Corollary 3.4.6. *Let*

$$T := \frac{f^3(-q^5)}{f(-q)f(-q^3)f(-q^{15})G(q)G^3(q^3)}.$$

Then

$$m - n + 1 = mn + 1 = \frac{m^2+m+1}{m+1} = \frac{n^2-n+1}{1-n} = 1 + uv^3 = T,$$

$$m^2 + m + 1 = \frac{Tm^2}{uv^3},$$

and

$$n^2 - n + 1 = \frac{Tn^2}{uv^3}.$$

3.5 Identities with Cubes of the Göllnitz–Gordon Functions

In this section, we derive analogues of Theorem 3.3.1 and Corollary 3.3.2 for the Göllnitz–Gordon functions. Our methods are similar to those used in Section 3.3. We record our results as Theorem 3.5.1.

Theorem 3.5.1. *We have*

$$\text{(i)} \qquad S^3(q)T(q^3) - T^3(q)S(q^3) = 3q\frac{f^3(-q^{24})}{f^3(-q^8)} \cdot \frac{\psi(-q^2)\phi(-q^4)}{\psi(-q)\psi(-q^3)}S(q)T(q),$$

$$\text{(ii)} \quad S^3(q^3)S(q) + q^5T^3(q^3)T(q) = \frac{f^3(-q^8)}{f^3(-q^{24})} \cdot \frac{\psi(-q^6)\phi(-q^{12})}{\psi(-q)\psi(-q^3)}S(q^3)T(q^3),$$

$$\text{(iii)} \quad \frac{S^3(q)T(q^3) - T^3(q)S(q^3)}{S^3(q^3)S(q) + q^5T^3(q^3)T(q)} = 3q\frac{f^4(-q^{24})}{f^4(-q^8)} \cdot \frac{\chi(q)}{\chi(q^3)}.$$

We offer two proofs of (i).

First Proof of (i). Recall the notation of Theorem 3.2.5. By (3.2.23) and Theorem 3.2.5,

$$\begin{aligned}
\frac{T(q^3)}{S(q^3)} - \frac{T^3(q)}{S^3(q)} &= \frac{f(-q^3,-q^{21})}{f(-q^9,-q^{15})} - \left(\frac{f(-q,-q^7)}{f(-q^3,-q^5)}\right)^3 \\
&= \frac{A^3-B^3}{C^3+D^3} - \left(\frac{A-B}{C+D}\right)^3 \\
&= \frac{(A^3-B^3)(C+D)^3 - (A-B)^3(C^3+D^3)}{(C^3+D^3)(C+D)^3} \\
&= \frac{3(A-B)(C+D)(AD+BC)(AC+BD)}{(C^3+D^3)(C+D)^3} \\
&= 3q\frac{f(q)f(-q^3,-q^{21})f^2(-q,-q^7)f(-q^4)f^2(-q^{24})\psi(q^3)}{f(-q^3,-q^5)f(-q^2,-q^6)f(-q^6,-q^{18})f^4(-q^8)f(-q^{12})}.
\end{aligned} \tag{3.5.1}$$

Now, by (3.1.1), (3.1.2), and (1.0.6),

$$S(q) = \frac{f^2(-q^8)}{f(-q,-q^7)f(-q^4)} \qquad \text{and} \qquad T(q) = \frac{f^2(-q^8)}{f(-q^3,-q^5)f(-q^4)}. \tag{3.5.2}$$

Employing (1.0.8), Lemma 1.0.1, and (3.5.2), we find that the last expression in (3.5.1) is equal to

$$\begin{aligned}
&3q\frac{f(q)f(-q^3,-q^{21})f^2(-q,-q^7)f(-q^4)f^2(-q^{24})\psi(q^3)}{f(-q^3,-q^5)\psi(-q^2)\psi(-q^6)f^4(-q^8)f(-q^{12})} \\
&= 3q\frac{f(-q^3,-q^{21})f^2(-q,-q^7)f(-q^4)f^2(-q^{24})}{f(-q^3,-q^5)f^4(-q^8)f(-q^{12})} \cdot \frac{f^3(-q^2)}{f(-q)f(-q^4)} \cdot \frac{f^2(-q^6)}{f(-q^3)} \\
&\quad \times \frac{f(-q^4)}{f(-q^2)f(-q^8)} \cdot \frac{f(-q^{12})}{f(-q^6)f(-q^{24})} \\
&= 3q\left[\frac{f(-q^3,-q^{21})f(-q^{12})}{f^2(-q^{24})}\right] \cdot \left[\frac{f^2(-q,-q^7)f^2(-q^4)}{f^4(-q^8)}\right] \cdot \left[\frac{f^2(-q^8)}{f(-q^3,-q^5)f(-q^4)}\right] \\
&\quad \times \left[\frac{f^3(-q^{24})}{f^3(-q^8)}\right] \cdot \left[\frac{f(-q^2)f(-q^8)}{f(-q^4)}\right] \cdot \left[\frac{f^2(-q^4)}{f(-q^8)}\right] \cdot \left[\frac{f(-q^2)}{f(-q)f(-q^4)}\right]
\end{aligned}$$

$$\times \left[\frac{f(-q^6)}{f(-q^3)f(-q^{12})}\right]$$
$$= 3q\frac{T(q)\psi(-q^2)\phi(-q^4)}{S(q^3)S^2(q)\psi(-q)\psi(-q^3)}\frac{f^3(-q^{24})}{f^3(-q^8)}. \tag{3.5.3}$$

Combining (3.5.1) and (3.5.3), we deduce that

$$\frac{T(q^3)}{S(q^3)} - \frac{T^3(q)}{S^3(q)} = 3q\frac{f^3(-q^{24})}{f^3(-q^8)}\frac{\psi(-q^2)\phi(-q^4)}{S(q^3)S^2(q)\psi(-q)\psi(-q^3)}T(q). \tag{3.5.4}$$

Multiplying (3.5.4) by $S^3(q)S(q^3)$, we complete the proof. □

Second Proof of (i). Applying (3.2.23), we rewrite (i) in the equivalent form

$$\begin{aligned} &f^3(-q^3,-q^5)f(-q^3,-q^{21}) - f^3(-q,-q^7)f(-q^9,-q^{15}) \\ &= 3q\frac{f^3(-q^{24})}{f^3(-q^8)}\psi(-q^2)\phi(-q^4)f(-q^3,-q^5)f(-q,-q^7). \end{aligned} \tag{3.5.5}$$

Set $\omega = e^{2\pi i/3}$. By (3.2.16), first with $a=-q, \quad b=-q^7$; and second with $a=-q^3, \quad b=-q^5$, we conclude that

$$f(-q^3,-q^{21}) = \frac{f(-q^{24})}{f^3(-q^8)}f(-\omega q,-\omega^2q^7)f(-\omega^2q,-\omega q^7)f(-q,-q^7) \tag{3.5.6}$$

and

$$f(-q^9,-q^{15}) = \frac{f(-q^{24})}{f^3(-q^8)}f(-\omega q^3,-\omega^2q^5)f(-\omega^2q^3,-\omega q^5)f(-q^3,-q^5). \tag{3.5.7}$$

Substituting (3.5.6) and (3.5.7) into (3.5.5) and simplifying, we deduce that (i) is equivalent to

$$\begin{aligned} &f^2(-q^3,-q^5)f(-\omega q,-\omega^2q^7)f(-\omega^2q,-\omega q^7) \\ &- f^2(-q,-q^7)f(-\omega q^3,-\omega^2q^5)f(-\omega^2q^3,-\omega q^5) = 3qf^2(-q^{24})\psi(-q^2)\phi(-q^4). \end{aligned} \tag{3.5.8}$$

We prove (3.5.8). Applying Lemma 1.0.2, first with $a=-\omega q,\ b=-\omega^2q^7,\ c=-q^3,\ d=-q^5$; secondly with $a=-\omega^2q,\ b=-\omega q^7,\ c=-q^3,\ d=-q^5$; thirdly with $a=-q,\ b=-q^7,\ c=-\omega q^3,\ d=-\omega^2q^5$; and fourthly with $a=-q,\ b=-q^7,\ c=-\omega^2q^3,\ d=-\omega q^5$, we deduce, respectively, that

$$\begin{aligned} f(-\omega q,-\omega^2q^7)f(-q^3,-q^5) =&f(\omega q^4,\omega^2q^{12})f(\omega q^6,\omega^2q^{10}) \\ &- \omega qf(\omega^2q^4,\omega q^{12})f(\omega^2q^2,\omega q^{14}), \end{aligned} \tag{3.5.9}$$
$$\begin{aligned} f(-\omega^2q,-\omega q^7)f(-q^3,-q^5) =&f(\omega^2q^4,\omega q^{12})f(\omega^2q^6,\omega q^{10}) \\ &- \omega^2qf(\omega q^4,\omega^2q^{12})f(\omega q^2,\omega^2q^{14}), \end{aligned} \tag{3.5.10}$$
$$\begin{aligned} f(-q,-q^7)f(-\omega q^3,-\omega^2q^5) =&f(\omega q^4,\omega^2q^{12})f(\omega^2q^6,\omega q^{10}) \\ &- qf(\omega^2q^4,\omega q^{12})f(\omega q^2,\omega^2q^{14}), \end{aligned} \tag{3.5.11}$$

and

$$\begin{aligned} f(-q,-q^7)f(-\omega^2q^3,-\omega q^5) =&f(\omega^2q^4,\omega q^{12})f(\omega q^6,\omega^2q^{10}) \\ &- qf(\omega q^4,\omega^2q^{12})f(\omega^2q^2,\omega q^{14}). \end{aligned} \tag{3.5.12}$$

Substitute (3.5.9)–(3.5.12) into (3.5.8). Expanding out and then simplifying, we conclude that (3.5.8) is equivalent to

$$\begin{aligned}
&f^2(\omega q^4,\omega^2 q^{12})f(\omega^2 q^6,\omega q^{10})f(\omega^2 q^2,\omega q^{14})\\
&+f^2(\omega^2 q^4,\omega q^{12})f(\omega q^6,\omega^2 q^{10})f(\omega q^2,\omega^2 q^{14})\\
&-\omega^2 f^2(\omega q^4,\omega^2 q^{12})f(\omega q^6,\omega^2 q^{10})f(\omega q^2,\omega^2 q^{14})\\
&-\omega f^2(\omega^2 q^4,\omega q^{12})f(\omega^2 q^6,\omega q^{10})f(\omega^2 q^2,\omega q^{14})=3f^2(-q^{24})\psi(-q^2)\phi(-q^4).
\end{aligned}\tag{3.5.13}$$

Next, we apply (3.3.15), first with $a=\omega q^4$, $b=\omega^2q^{12}$; secondly with $a=\omega q^6$, $b=\omega^2q^{10}$; and thirdly with $a=\omega q^2$, $b=\omega^2q^{14}$ in order to deduce, respectively, that

$$f(\omega q^4,\omega^2 q^{12})=f(q^{60},q^{84})+\omega q^4 f(q^{36},q^{108})+\omega^2 q^{12}f(q^{12},q^{132}),\tag{3.5.14}$$

$$f(\omega q^6,\omega^2 q^{10})=f(q^{66},q^{78})+\omega q^6 f(q^{30},q^{114})+\omega^2 q^{10}f(q^{18},q^{126}),\tag{3.5.15}$$

and

$$f(\omega q^2,\omega^2 q^{14})=f(q^{54},q^{90})+\omega q^2 f(q^{42},q^{102})+\omega^2 q^{14}f(q^{6},q^{138}).\tag{3.5.16}$$

Replacing ω with ω^2, we obtain one further identity from each of (3.5.14)–(3.5.16). Substitute the six identities thus found into (3.5.13), expand out the resulting expressions, and simplify algebraically with the help of MAPLE. We therefore find that (3.5.8) is equivalent to

$$\begin{aligned}
&\Big[q^{20}f(q^{30},q^{114})f(q^6,q^{138})-q^{24}f(q^6,q^{138})f(q^{18},q^{126})\\
&\qquad+q^{12}f(q^{18},q^{126})f(q^{42},q^{102})-q^2f(q^{42},q^{102})f(q^{66},q^{78})\\
&\qquad+f(q^{66},q^{78})f(q^{54},q^{90})-q^6f(q^{54},q^{90})f(q^{30},q^{114})\Big]\\
&\times\Big[f^2(q^{60},q^{84})-q^{24}f^2(q^{12},q^{132})+2q^{16}f(q^{36},q^{108})f(q^{12},q^{132})\\
&\qquad-2q^4f(q^{36},q^{108})f(q^{60},q^{84})\Big]=f^2(-q^{24})\psi(-q^2)\phi(-q^4).
\end{aligned}\tag{3.5.17}$$

With the help of Theorem 3.2.4(i), (iv), we readily deduce the truth of (3.5.17), and hence also of (3.5.8). This completes the proof. □

Proof of (ii). Employing (3.2.23), we rewrite (ii) in the equivalent form

$$\begin{aligned}
&f^3(-q^9,-q^{15})f(-q^3,-q^5)+q^5f^3(-q^3,-q^{21})f(-q,-q^7)\\
&\qquad=\frac{f^3(-q^8)}{f^3(-q^{24})}\psi(-q^6)\phi(-q^{12})f(-q^9,-q^{15})f(-q^3,-q^{21}).
\end{aligned}\tag{3.5.18}$$

We prove (3.5.18) by transforming the left-hand side into the right-hand side. Employing (1.0.6) and (1.0.9), we readily deduce that

$$f(-q^3,-q^5)=f(-q^{11},-q^{13})f(-q^5,-q^{19})f(-q^3,-q^{21})\frac{f(-q^8)}{f^3(-q^{24})}\tag{3.5.19}$$

and

$$f(-q,-q^7)=f(-q,-q^{23})f(-q^9,-q^{15})f(-q^7,-q^{17})\frac{f(-q^8)}{f^3(-q^{24})}. \tag{3.5.20}$$

Applying (3.5.19) and (3.5.20), we find that the left-hand side of (3.5.18) is equal to

$$\begin{aligned}&\frac{f(-q^8)}{f^3(-q^{24})}f(-q^9,-q^{15})f(-q^3,-q^{21})\\&\times\Big\{f^2(-q^9,-q^{15})f(-q^{11},-q^{13})f(-q^5,-q^{19})\\&\qquad+q^5f^2(-q^3,-q^{21})f(-q,-q^{23})f(-q^7,-q^{17})\Big\}.\end{aligned} \tag{3.5.21}$$

Next, we apply Lemma 1.0.2 four times: first with $a=-q^9,\ b=-q^{15},\ c=-q^{11}$, $d=-q^{13}$; secondly with $a=-q^9,\ b=-q^{15},\ c=-q^5,\ d=-q^{19}$; thirdly with $a=-q^3,\ b=-q^{21},\ c=-q,\ d=-q^{23}$; and fourthly with $a=-a^3,\ b=-q^{21}$, $c=-q^7$, and $d=-q^{17}$, in order to deduce, respectively, that

$$f(-q^9,-q^{15})f(-q^{11},-q^{13})=f(q^{20},q^{28})f(q^{22},q^{26})-q^9f(q^4,q^{44})f(q^2,q^{46}), \tag{3.5.22}$$

$$f(-q^9,-q^{15})f(-q^5,-q^{19})=f(q^{14},q^{34})f(q^{20},q^{28})-q^5f(q^4,q^{44})f(q^{10},q^{38}), \tag{3.5.23}$$

$$f(-q^3,-q^{21})f(-q,-q^{23})=f(q^4,q^{44})f(q^{22},q^{26})-qf(q^2,q^{46})f(q^{20},q^{28}), \tag{3.5.24}$$

and

$$f(-q^3,-q^{21})f(-q^7,-q^{17})=f(q^{10},q^{38})f(q^{20},q^{28})-q^3f(q^4,q^{44})f(q^{14},q^{34}). \tag{3.5.25}$$

Here (1.0.5) was used to simplify (3.5.23) and (3.5.24). Substituting (3.5.22)–(3.5.25) into (3.5.21), we find that the left-hand side of (3.5.18) is equal to

$$\begin{aligned}&\frac{f(-q^8)}{f^3(-q^{24})}f(-q^9,-q^{15})f(-q^3,-q^{21})\\&\times\Big\{\left[f(q^{20},q^{28})f(q^{22},q^{26})-q^9f(q^4,q^{44})f(q^2,q^{46})\right]\\&\qquad\times\left[f(q^{14},q^{34})f(q^{20},q^{28})-q^5f(q^4,q^{44})f(q^{10},q^{38})\right]\\&\quad+q^5\left[f(q^4,q^{44})f(q^{22},q^{26})-qf(q^2,q^{46})f(q^{20},q^{28})\right]\\&\qquad\times\left[f(q^{10},q^{38})f(q^{20},q^{28})-q^3f(q^4,q^{44})f(q^{14},q^{34})\right]\Big\},\end{aligned}$$

which is algebraically equivalent to

$$\begin{aligned}&\frac{f(-q^8)}{f^3(-q^{24})}f(-q^9,-q^{15})f(-q^3,-q^{21})\Big\{\left[f^2(q^{20},q^{28})-q^8f^2(q^4,q^{44})\right]\\&\quad\times\left[f(q^{22},q^{26})f(q^{14},q^{34})-q^6f(q^2,q^{46})f(q^{10},q^{38})\right]\Big\}.\end{aligned} \tag{3.5.26}$$

Next, we apply Lemma 1.0.2 two more times, first with $a=-q^8,\ b=-q^{16}$,

$c=d=-q^{12}$; and secondly with $a=-q^8,\ b=-q^{16},\ c=-q^6,\ d=-q^{18}$, along with one application of (1.0.5), in order to deduce that

$$f(-q^8,-q^{16})f(-q^{12},-q^{12})=f^2(q^{20},q^{28})-q^8f^2(q^4,q^{44}) \tag{3.5.27}$$

and

$$f(-q^8,-q^{16})f(-q^6,-q^{18})=f(q^{22},q^{26})f(q^{14},q^{34})-q^6f(q^2,q^{46})f(q^{10},q^{38}). \tag{3.5.28}$$

Substituting (3.5.27) and (3.5.28) into (3.5.26), we deduce that the left-hand side of (3.5.18) is equal to

$$\begin{aligned}&\frac{f(-q^8)}{f^3(-q^{24})}f(-q^9,-q^{15})f(-q^3,-q^{21})\\&\qquad\times\Big[f(-q^8,-q^{16})f(-q^{12},-q^{12})\Big]\times\Big[f(-q^8,-q^{16})f(-q^6,-q^{18})\Big].\end{aligned} \tag{3.5.29}$$

Employing (1.0.7)–(1.0.9) in (3.5.29), we conclude, finally, that the left-hand side of (3.5.18) is equal to

$$\frac{f^3(-q^8)}{f^3(-q^{24})}f(-q^9,-q^{15})f(-q^3,-q^{21})\psi(-q^6)\phi(-q^{12}). \tag{3.5.30}$$

This proves (3.5.18), and hence completes the proof. □

Proof of (iii). Divide (i) by (ii). On the right-hand side, apply (3.2.24) and Lemma 1.0.1 to complete the proof. □

3.6 Further Relations for the Göllnitz–Gordon Functions

In this section, we develop four related identities for the Göllnitz–Gordon functions, each of which connects these functions at the arguments q and q^3. New proofs for each relation are given, and one of our identities is new. In Section 3.7, we combine the results of this section with those in Section 3.5 in order to give applications to the Ramanujan–Göllnitz–Gordon continued fraction.

Theorem 3.6.1. *We have*

(i) $$S(q^3)T(q)+qS(q)T(q^3)=\frac{\phi(-q^4)\psi(-q^6)}{\psi(-q)\psi(-q^3)},$$

(ii) $$S(q^3)S(q)-q^2T(q^3)T(q)=\frac{\psi(-q^2)\phi(-q^{12})}{\psi(-q)\psi(-q^3)},$$

(iii) $$S(q^3)T(q)-qS(q)T(q^3)=\frac{f(-q)f(-q^{12})}{f(-q^3)f(-q^4)},$$

(iv) $$S(q^3)S(q)+q^2T(q^3)T(q)=\frac{f(-q^3)f(-q^4)}{f(-q)f(-q^{12})}.$$

Identities (i) and (iii) were first discovered by S. Robins [80], who utilized the theory of modular forms. Using techniques of Bressoud, Huang [58] proved both (iii) and (iv), and furthermore offered partition-theoretic interpretations of those two identities. Subsequently, Baruah, Bora, and Saikia [13] discovered new proofs of (iii) and (iv). Identity (ii) is new.

We present two proofs for each identity. Identities (i) and (iii) are equivalent, as our proofs will show; similarly, (ii) and (iv) are equivalent. In the first proof of (i) and (iii), we derive the two relations simultaneously using theta function identities. In the second proof of (i) and (iii), we employ the theory of modular equations to show directly that each identity implies the other. A similar pair of proofs is then offered for (ii) and (iv).

First proof of (i),(iii). Applying (3.2.23) and Lemma 2.3.3, we rewrite (i) and (iii), respectively, in the equivalent forms

$$f(-q^9,-q^{15})f(-q,-q^7)+qf(-q^3,-q^5)f(-q^3,-q^{21})=\phi(-q^4)\psi(-q^6), \tag{3.6.1}$$

$$f(-q^9,-q^{15})f(-q,-q^7)-qf(-q^3,-q^5)f(-q^3,-q^{21})=\phi(-q)\psi(q^6). \tag{3.6.2}$$

Adding, respectively subtracting, (3.6.1) and (3.6.2), we obtain

$$2f(-q^9,-q^{15})f(-q,-q^7)=\phi(-q^4)\psi(-q^6)+\phi(-q)\psi(q^6), \tag{3.6.3}$$

$$2qf(-q^3,-q^5)f(-q^3,-q^{21})=\phi(-q^4)\psi(-q^6)-\phi(-q)\psi(q^6). \tag{3.6.4}$$

It is clear that (3.6.3) and (3.6.4) together imply both (3.6.1) and (3.6.2), and hence also (i) and (iii). Thus, it suffices to prove (3.6.3) and (3.6.4). To that end, we employ (1.0.7), along with (1.0.15) with $a=b=-q$ and $n=2$, in order to see that

$$\phi(-q)=f(-q,-q)=f(q^4,q^4)-qf(1,q^8). \tag{3.6.5}$$

Alternatively, one can show (3.6.5) directly by considering the even-odd dissection of the sum in (1.0.7).

Combining (3.6.5) with (1.0.8), we find that

$$\phi(-q)\psi(q^6)=f(q^4,q^4)f(q^6,q^{18})-qf(1,q^8)f(q^6,q^{18}). \tag{3.6.6}$$

Observe that, by (1.0.4),

$$f(-1,-q^8)f(-q^6,-q^{18})=0. \tag{3.6.7}$$

Thus, by (1.0.7), (1.0.8), (3.6.6), and (3.6.7),

$$\begin{aligned}\phi(-q^4)\psi(-q^6)+\phi(-q)\psi(q^6)&=f(-q^4,-q^4)f(-q^6,-q^{18})+f(q^4,q^4)f(q^6,q^{18})\\&\quad-qf(1,q^8)f(q^6,q^{18})+qf(-1,-q^8)f(-q^6,-q^{18}).\end{aligned} \tag{3.6.8}$$

Next, apply Theorem 1.0.3 with the parameters $a=b=q^4$, $c=q^6$, $d=q^{18}$, $\alpha=1$, $\beta=3$, $\epsilon_1=\epsilon_2=0$, and $m=4$. We consequently find that

$$\begin{aligned}f(q^4,q^4)f(q^6,q^{18})&=f(q^{10},q^{22})f(q^{42},q^{54})+q^6f(q^{-2},q^{34})f(q^{30},q^{66})\\&\quad+q^{36}f(q^{-26},q^{58})f(q^6,q^{90})+q^{90}f(q^{-50},q^{82})f(q^{-18},q^{114})\\&=f(q^{10},q^{22})f(q^{42},q^{54})+q^4f(q^2,q^{30})f(q^{30},q^{66})\end{aligned}$$

$$+ q^{10} f(q^6, q^{26}) f(q^6, q^{90}) + q^4 f(q^{14}, q^{18}) f(q^{18}, q^{78}), \tag{3.6.9}$$

where we applied (1.0.5) four times in the last equality. Applying Theorem 1.0.3 with the same set of parameters, except now with $\epsilon_1 = \epsilon_2 = 1$, we similarly deduce that

$$\begin{aligned} f(-q^4, -q^4) f(-q^6, -q^{18}) = &f(q^{10}, q^{22}) f(q^{42}, q^{54}) - q^4 f(q^2, q^{30}) f(q^{30}, q^{66}) \\ &+ q^{10} f(q^6, q^{26}) f(q^6, q^{90}) - q^4 f(q^{14}, q^{18}) f(q^{18}, q^{78}). \end{aligned} \tag{3.6.10}$$

By a third application of Theorem 1.0.3, this time with the parameters $a = 1$, $b = q^8$, $c = q^6$, $d = q^{18}$, $\alpha = 1$, $\beta = 3$, $\epsilon_1 = \epsilon_2 = 0$, and $m = 4$, we find that

$$\begin{aligned} f(1, q^8) f(q^6, q^{18}) &= f(q^{14}, q^{18}) f(q^{30}, q^{66}) + q^6 f(q^{-6}, q^{38}) f(q^{42}, q^{54}) \\ &\quad + q^{36} f(q^{-30}, q^{62}) f(q^{18}, q^{78}) + q^{90} f(q^{-54}, q^{86}) f(q^{-6}, q^{102}) \\ &= f(q^{14}, q^{18}) f(q^{30}, q^{66}) + f(q^6, q^{26}) f(q^{42}, q^{54}) \\ &\quad + q^6 f(q^2, q^{30}) f(q^{18}, q^{78}) + q^8 f(q^{10}, q^{22}) f(q^6, q^{90}), \end{aligned} \tag{3.6.11}$$

where we applied (1.0.5) four times in the last equality. Employing Theorem 1.0.3 with the same set of parameters, but now with $\epsilon_1 = \epsilon_2 = 1$, we deduce after simplifying that

$$\begin{aligned} f(-1, -q^8) f(-q^6, -q^{18}) = &f(q^{14}, q^{18}) f(q^{30}, q^{66}) - f(q^6, q^{26}) f(q^{42}, q^{54}) \\ &+ q^6 f(q^2, q^{30}) f(q^{18}, q^{78}) - q^8 f(q^{10}, q^{22}) f(q^6, q^{90}). \end{aligned} \tag{3.6.12}$$

Combining (3.6.8)–(3.6.12), we conclude that

$$\begin{aligned} &\phi(-q^4)\psi(-q^6) + \phi(-q)\psi(q^6) \\ &\quad = 2f(q^{10}, q^{22}) f(q^{42}, q^{54}) - 2q f(q^6, q^{26}) f(q^{42}, q^{54}) \\ &\quad\quad - 2q^9 f(q^{10}, q^{22}) f(q^6, q^{90}) + 2q^{10} f(q^6, q^{26}) f(q^6, q^{90}) \\ &\quad = 2\left(f(q^{42}, q^{54}) - q^9 f(q^6, q^{90})\right)\left(f(q^{10}, q^{22}) - q f(q^6, q^{26})\right). \end{aligned} \tag{3.6.13}$$

By (1.0.15) with $a = -q$, $b = -q^7$, $n = 2$, and with $a = -q^3$, $b = -q^5$, $n = 2$, we deduce, respectively, that

$$f(-q, -q^7) = f(q^{10}, q^{22}) - q f(q^6, q^{26}), \tag{3.6.14}$$
$$f(-q^3, -q^5) = f(q^{14}, q^{18}) - q^3 f(q^2, q^{30}). \tag{3.6.15}$$

Replacing q by q^3 in each of (3.6.14), (3.6.15), we find, respectively, that

$$f(-q^3, -q^{21}) = f(q^{30}, q^{66}) - q^3 f(q^{18}, q^{78}), \tag{3.6.16}$$
$$f(-q^9, -q^{15}) = f(q^{42}, q^{54}) - q^9 f(q^6, q^{90}). \tag{3.6.17}$$

Thus, employing (3.6.17) and (3.6.14) in (3.6.13), we readily deduce the truth of (3.6.3).

Similarly, to prove (3.6.4), observe that, by (1.0.7), (1.0.8), (3.6.6), and (3.6.7),

$$\phi(-q^4)\psi(-q^6) - \phi(-q)\psi(q^6) = f(-q^4, -q^4) f(-q^6, -q^{18}) - f(q^4, q^4) f(q^6, q^{18})$$

$$+\,qf(1,q^8)f(q^6,q^{18})+qf(-1,-q^8)f(-q^6,-q^{18}). \tag{3.6.18}$$

Applying (3.6.9)–(3.6.12), followed by (3.6.15) and (3.6.16), we conclude from (3.6.18) that

$$\begin{aligned}
&\phi(-q^4)\psi(-q^6)-\phi(-q)\psi(q^6)\\
&\qquad=-2q^4f(q^2,q^{30})f(q^{30},q^{66})-2q^4f(q^{14},q^{18})f(q^{18},q^{78})\\
&\qquad\quad+2qf(q^{14},q^{18})f(q^{30},q^{66})+2q^7f(q^2,q^{30})f(q^{18},q^{78})\\
&\qquad=2q\left(f(q^{14},q^{18})-q^3f(q^2,q^{30})\right)\left(f(q^{30},q^{66})-q^3f(q^{18},q^{78})\right)\\
&\qquad=2qf(-q^3,-q^5)f(-q^3,-q^{21}).
\end{aligned}$$

Thus (3.6.4) is proved, and we complete the first proof of (i) and (iii). □

Second proof of (i),(iii). We begin with the trivial relation

$$\begin{aligned}
&\left[S(q^3)T(q)+qS(q)T(q^3)\right]^2\\
&=\left[S(q^3)T(q)-qS(q)T(q^3)\right]^2+4qS(q^3)T(q^3)S(q)T(q).
\end{aligned} \tag{3.6.19}$$

Apply (3.2.24) twice to the last term in (3.6.19), replacing q by q^3 in the second application. Substitute also the right-hand sides of Theorem 3.6.1(i),(iii) into (3.6.19), using Lemma 1.0.1 to rewrite (i) in terms of $f(-q)$. We therefore arrive at the identity

$$\begin{aligned}
&\frac{f^2(-q^2)f^2(-q^4)f^4(-q^6)f^2(-q^{24})}{f^2(-q)f^2(-q^3)f^2(-q^8)f^4(-q^{12})}\\
&\qquad=\frac{f^2(-q)f^2(-q^{12})}{f^2(-q^3)f^2(-q^4)}+4q\frac{f(-q^2)f(-q^6)f^2(-q^8)f^2(-q^{24})}{f(-q)f(-q^3)f^2(-q^4)f^2(-q^{12})}.
\end{aligned}$$

A further slight modification using Lemma 1.0.1 yields

$$\frac{\psi(q^{12})\psi^2(q^3)f(-q^4)}{\psi(q^4)\chi^2(-q)f^3(-q^{12})}=\frac{f^2(-q)f^2(-q^{12})}{f^2(-q^3)f^2(-q^4)}+4q\frac{\psi(q^4)\psi(q^{12})f(-q^2)f(-q^6)}{f(-q)f(-q^3)f(-q^4)f(-q^{12})}. \tag{3.6.20}$$

It is clear from (3.6.19) and (3.6.20) that identity (i) holds if and only if identity (iii) holds. Thus, it remains only to give a direct proof of (3.6.20).

In order to prove (3.6.20), we employ the theory of modular equations. Suppose that β has degree three over α. Then, converting via Lemma 1.0.4 and simplifying, we see that (3.6.20) is equivalent to the modular equation

$$\begin{aligned}
\left(\frac{\beta\,(1-\alpha)}{\alpha\,(1-\beta)}\right)^{1/4}&=\frac{(\alpha/\beta)^{1/4}\left(1-\sqrt{1-\beta}\right)^{1/2}}{\{(1-\alpha)\,(1-\beta)\}^{1/8}\left(1-\sqrt{1-\alpha}\right)^{1/2}}\\
&\qquad-\frac{\left(1-\sqrt{1-\alpha}\right)^{1/2}\left(1-\sqrt{1-\beta}\right)^{1/2}}{\{\alpha\beta\,(1-\alpha)\,(1-\beta)\}^{1/8}}.
\end{aligned} \tag{3.6.21}$$

Squaring both sides of (3.6.21), we obtain

$$\left(\frac{\beta\,(1-\alpha)}{\alpha\,(1-\beta)}\right)^{1/2}=\frac{(\alpha/\beta)^{1/2}\left(1-\sqrt{1-\beta}\right)}{\{(1-\alpha)\,(1-\beta)\}^{1/4}\left(1-\sqrt{1-\alpha}\right)}$$

$$+\frac{\left(1-\sqrt{1-\alpha}\right)\left(1-\sqrt{1-\beta}\right)}{(\alpha\beta)^{1/4}\left\{(1-\alpha)\,(1-\beta)\right\}^{1/4}}-2\frac{\alpha^{1/8}\left(1-\sqrt{1-\beta}\right)}{\beta^{3/8}\left\{(1-\alpha)\,(1-\beta)\right\}^{1/4}}. \tag{3.6.22}$$

Moving all terms in (3.6.22) to the right-hand side of the equality, multiplying by $\{(1-\alpha)\,(1-\beta)\}^{1/4}$, and using the elementary identity $1/\left(1-\sqrt{1-\alpha}\right)=\left(1+\sqrt{1-\alpha}\right)/\alpha$, we find that

$$\begin{aligned}0=&-\frac{(1-\alpha)^{3/4}}{(1-\beta)^{1/4}}\left(\frac{\beta}{\alpha}\right)^{1/2}+\frac{\left(1-\sqrt{1-\beta}\right)\left(1+\sqrt{1-\alpha}\right)}{(\alpha\beta)^{1/2}}\\&+\frac{\left(1-\sqrt{1-\alpha}\right)\left(1-\sqrt{1-\beta}\right)}{(\alpha\beta)^{1/4}}-2\frac{\alpha^{1/8}}{\beta^{3/8}}\left(1-\sqrt{1-\beta}\right).\end{aligned} \tag{3.6.23}$$

We now parameterize (3.6.23) in terms of the multiplier of degree 3, namely,

$$m=\frac{\phi^2(q)}{\phi^2(q^3)}.$$

By [14, p. 233, Equations (5.2),(5.5)],

$$\alpha=\frac{(m-1)\,(3+m)^3}{16m^3},\qquad \beta=\frac{(m-1)^3\,(3+m)}{16m}, \tag{3.6.24}$$

$$1-\alpha=\frac{(m+1)\,(3-m)^3}{16m^3},\qquad 1-\beta=\frac{(m+1)^3\,(3-m)}{16m}. \tag{3.6.25}$$

We consequently deduce the relations

$$\left(\frac{\beta}{\alpha}\right)^{1/2}=\frac{m\,(m-1)}{3+m},\qquad \frac{(1-\alpha)^{3/4}}{(1-\beta)^{1/4}}=\frac{(3-m)^2}{4m^2}, \tag{3.6.26}$$

$$(\alpha\beta)^{1/4}=\frac{(m+3)\,(m-1)}{4m},\qquad \frac{\alpha^{1/8}}{\beta^{3/8}}=\frac{2}{m-1}, \tag{3.6.27}$$

and

$$\sqrt{1-\alpha}=\frac{3-m}{m\,(m+1)}\sqrt{1-\beta}. \tag{3.6.28}$$

Define

$$w:=\sqrt{1-\beta}. \tag{3.6.29}$$

Employing (3.6.24)–(3.6.29), we rewrite (3.6.23) in the form

$$\begin{aligned}0=&-\frac{(3-m)^2}{4m^2}\cdot\frac{m\,(m-1)}{3+m}+\frac{16m^2}{(m+3)^2\,(m-1)^2}\,(1-w)\left(1+\frac{3-m}{m\,(m+1)}w\right)\\&+\frac{4m}{(m+3)\,(m-1)}\left(1-\frac{3-m}{m\,(m+1)}w\right)(1-w)-\frac{4(1-w)}{m-1}.\end{aligned} \tag{3.6.30}$$

Since

$$w^2=1-\beta=\frac{(m+1)^3\,(3-m)}{16m}, \tag{3.6.31}$$

by (3.6.29) and (3.6.25), we readily see that the right-hand side of (3.6.30) can be expressed as a linear polynomial $aw+b$ in w with coefficients a, b, that are rational functions of the multiplier m. After some algebra, we conclude that $a = b = 0$. Hence, (3.6.30) is true. This proves (3.6.20), and thus completes the proof. $\square$

First Proof of (ii),(iv). The first proof of (ii) and (iv) is analogous to the first proof of (i) and (iii), and so we omit the details. $\square$

Second Proof of (ii),(iv). The proof is similar to the second proof of (i) and (iii), and so we provide only a brief sketch. Beginning with (ii) and (iv), we find that the theta function analogue of (3.6.20) that we must prove is

$$\begin{aligned}\frac{f^2(-q^3)f^2(-q^4)}{f^2(-q)f^2(-q^{12})} &= \frac{\psi(q^4)f^2(-q^2)f(-q^{12})}{\chi^2(-q)\chi^2(-q^3)\psi(q^{12})f^3(-q^4)} \\ &\quad + 4q^2\frac{\psi(q^4)\psi(q^{12})f(-q^2)f(-q^6)}{f(-q)f(-q^3)f(-q^4)f(-q^{12})}.\end{aligned} \tag{3.6.32}$$

Applying Lemma 1.0.4, we convert (3.6.32) into the modular equation

$$\begin{aligned}\left(\frac{\alpha(1-\beta)}{\beta(1-\alpha)}\right)^{1/4} &= \frac{(\beta/\alpha)^{1/4}\left(1-\sqrt{1-\alpha}\right)^{1/2}}{\left(1-\sqrt{1-\beta}\right)^{1/2}\{(1-\alpha)(1-\beta)\}^{1/8}} \\ &\quad + \frac{\left(1-\sqrt{1-\alpha}\right)^{1/2}\left(1-\sqrt{1-\beta}\right)^{1/2}}{\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/8}}.\end{aligned} \tag{3.6.33}$$

Proceeding as in the proof of (3.6.21), we complete the proof. $\square$

3.7 Modular Relations for the Ramanujan–Göllnitz–Gordon Continued Fraction

In this section, we apply the results of the preceding two sections in order to prove modular relations for the Ramanujan–Göllnitz–Gordon continued fraction, $K(q)$.

The following theorem is new.

Theorem 3.7.1. *Let $u = K(q)$ and $v = K(q^3)$. Then*

(i) $$v - u^3 = 3q^{5/2}\frac{f^3(-q^{24})}{f^3(-q^8)}\frac{\psi(-q^2)\phi(-q^4)}{\psi(-q)\psi(-q^3)}\frac{T(q)}{S^2(q)S(q^3)},$$

(ii) $$1 + uv^3 = \frac{f^3(-q^8)}{f^3(-q^{24})}\frac{\psi(-q^6)\phi(-q^{12})}{\psi(-q)\psi(-q^3)}\frac{T(q^3)}{S(q)S^2(q^3)},$$

(iii) $$v + u = q^{1/2}\frac{\phi(-q^4)\psi(-q^6)}{\psi(-q)\psi(-q^3)}\frac{1}{S(q)S(q^3)},$$

(iv) $$1 - uv = \frac{\psi(-q^2)\phi(-q^{12})}{\psi(-q)\psi(-q^3)}\frac{1}{S(q)S(q^3)}.$$

Proof. Multiplying the identity in Theorem 3.5.1(i) by $q^{3/2}/(S^3(q)S(q^3))$ and applying (3.1.4), we deduce (i). By analogous arguments, (ii), (iii), and (iv) follow from (3.1.4) along with, respectively, Theorem 3.5.1(ii), Theorem 3.6.1 (i),(ii). $\square$

The next result, Theorem 3.7.2, is an analogue of Ramanujan's identity in Theorem 3.4.2. Chan and Huang [34] offered the first systematic study of modular relations for the Ramanujan–Göllnitz–Gordon continued fraction, and gave the first proof of Theorem 3.7.2. Using theta function identities, Vasuki and Srivatsa Kumar [96] offered a new proof, and found further relations for $K(q)$ as well. Recently, Cho, Koo, and Park [36] found a new proof as part of their study of modular relations for $K(q)$ from the viewpoint of the theory of modular forms.

Our proof, which uses Theorem 3.7.1, is the first to give identities for the four factors in parentheses.

Theorem 3.7.2. *Let u and v be as in Theorem 3.7.1. Then*

$$\left(v-u^3\right)\left(1+uv^3\right)=3uv\,(1-uv)\,(u+v)\,.$$

Proof. Applying Theorem 3.7.1 and (3.1.4), we conclude that

$$\begin{aligned}\left(v-u^3\right)\left(1+uv^3\right)&=3q^{5/2}\frac{\psi(-q^2)\phi(-q^4)\psi(-q^6)\phi(-q^{12})}{\psi^2(-q)\psi^2(-q^3)}\frac{T(q)T(q^3)}{S^3(q)S^3(q^3)}\\&=3\left(\frac{q^2T(q)T(q^3)}{S(q)S(q^3)}\right)\left(\frac{\psi(-q^2)\phi(-q^{12})}{\psi(-q)\psi(-q^3)}\frac{1}{S(q)S(q^3)}\right)\\&\quad\times\left(q^{1/2}\frac{\phi(-q^4)\psi(-q^6)}{\psi(-q)\psi(-q^3)}\frac{1}{S(q)S(q^3)}\right)\\&=3uv\,(1-uv)\,(u+v)\,.\end{aligned}$$

This completes the proof. □

The following identities are new.

Theorem 3.7.3. *Let u and v be as in Theorem 3.7.1. Then*

(i) $$\frac{v-u^3}{1+uv^3}=3q^{7/2}\frac{u}{v}\cdot\frac{f^6(-q^{24})}{f^6(-q^8)}\frac{\psi(-q^2)\phi(-q^4)}{\psi(-q^6)\phi(-q^{12})},$$

(ii) $$\frac{u^3}{v}+\frac{v}{u^3}=2+9q^2\frac{f^6(-q^{24})}{f^6(-q^8)}\cdot\frac{\psi^2(-q^2)\phi^2(-q^4)}{\psi^2(-q)\psi^2(-q^3)S(q)T(q)S(q^3)T(q^3)},$$

(iii) $$\frac{1}{uv^3}+uv^3=-2+\frac{f^6(-q^8)}{q^5f^6(-q^{24})}\cdot\frac{\psi^2(-q^6)\phi^2(-q^{12})}{\psi^2(-q)\psi^2(-q^3)S(q)T(q)S(q^3)T(q^3)}.$$

Proof. In Theorem 3.7.1, divide (i) by (ii). Simplifying with the help of (3.1.4), we complete the proof of (i). Identities (ii) and (iii) follow from parts (i) and (ii), respectively, of Theorem 3.7.1. The proofs are analogous to the proofs of Theorem 3.4.1(iv),(v); we omit the details. □

We remark that, with the use of Theorem 3.5.1, Theorem 3.6.1, (3.1.4), and Lemma 1.0.1, many further identities connecting $u=K(q)$ and $v=K(q^3)$ may be developed. We illustrate with one further theorem. The proofs are analogous to those of the preceding theorems, and so we omit the details.

Theorem 3.7.4. *With u and v as in Theorem 3.7.1, we have*

(i)
$$\frac{u+v}{1-uv}=\sqrt{q}\frac{\phi(-q^4)\psi(-q^6)}{\phi(-q^{12})\psi(-q^2)},$$

(ii)
$$\frac{u-v}{1+uv}=\sqrt{q}\frac{\phi(-q)\psi(q^6)}{\phi(-q^3)\psi(q^2)},$$

(iii)
$$\frac{u^2-v^2}{1-u^2v^2}=q\frac{\phi(-q)\psi(q^{12})}{\phi(-q^3)\psi(q^4)},$$

(iv)
$$\frac{u-v}{u+v}=\frac{\phi(-q)\phi(-q^{12})}{\phi(-q^4)\phi(-q^6)},$$

(v)
$$\frac{1-uv}{1+uv}=\frac{\phi(-q^2)\phi(-q^{12})}{\phi(-q^3)\phi(-q^4)}.$$

3.8 Applications to the Theory of Partitions

The identities in Theorems 3.3.1, 3.5.1, and 3.6.1 all have applications in the theory of partitions. To describe these, we require the notion of colored partitions. We say that a positive integer n has k colors if there are k copies of n available and all of them are viewed as distinct objects. Partitions of positive integers into parts with colors are called *colored partitions*. For example, if 1 is allowed to have two colors, say red (r) and green (g), then the colored partitions of 2 are $2, 1_r+1_r, 1_g+1_g, 1_r+1_g$. An important fact is that

$$\frac{1}{(q^r;q^s)_\infty^k}$$

is the generating function for the number of partitions of n, where all the parts are congruent to $r \pmod{s}$ and have k colors.

We introduce next notation and a lemma that is useful for extracting partition results from the modular relations that we consider. Let $p_e(n)$ denote the number of partitions of n into an even number of parts, and let $p_o(n) := p(n)-p_e(n)$ denote the number of partitions of n into an odd number of parts, where $p(n)$ is the ordinary partition function. Set $p(0)=1$.

Lemma 3.8.1. *The following identities hold.*

$$\frac{1}{(-q;q)_\infty}=1+\sum_{n=1}^{\infty}\left(p_e(n)-p_o(n)\right)q^n, \tag{3.8.1}$$

$$p_e(n)=\sum_{0\le j\le\sqrt{n}}(-1)^j p(n-j^2), \qquad n=1,2\dots. \tag{3.8.2}$$

Proof. For details, see [41, pp. 38–39, Equations (22.14), (22.21)]. □

For simplicity, in this section we employ the standard notation

$$(a_1,a_2,\dots,a_n;q^t)_\infty:=\prod_{j=1}^{n}(a_j;q^t)_\infty$$

and, for positive integers r and s with $r < s$,

$$(q^{r\pm}; q^s)_\infty := (q^r, q^{s-r}; q^s)_\infty.$$

We begin with applications of Theorem 3.3.1.

Theorem 3.8.2. *Let $p_1(n)$ denote the number of partitions of n into parts congruent to $\pm 1 \pmod 5$, where the parts congruent to $\pm 1, \pm 4 \pmod{15}$ have three colors and the parts congruent to $\pm 6 \pmod{15}$ have four colors.*

Let $p_2(n)$ denote the number of partitions of n into parts congruent to $\pm 2 \pmod 5$, where the parts congruent to $\pm 2, \pm 7 \pmod{15}$ have three colors and the parts congruent to $\pm 3 \pmod{15}$ have four colors.

Let $p_3(n)$ denote the number of partitions of n into parts not divisible by 15, and with parts congruent to $\pm 3, \pm 5, \pm 6 \pmod{15}$ having two colors.

Define $p_i(0) := 1$ for $i = 1, 2, 3$, and $p_3(n) := 0$ for $n < 0$. Then, for each nonnegative integer n,

$$p_1(n) - p_2(n) = 3p_3(n-1).$$

Proof. Using (2.1.2), (2.1.3), and (1.0.9), we write each of the functions appearing in Theorem 3.3.1(i) in terms of its product representation. Therefore,

$$\frac{1}{(q^{1\pm}; q^5)_\infty^3 (q^{6\pm}; q^{15})_\infty} - \frac{1}{(q^{2\pm}; q^5)_\infty^3 (q^{3\pm}; q^{15})_\infty} = \frac{3q(q^{15}; q^{15})_\infty^3}{(q; q)_\infty (q^3; q^3)_\infty (q^5; q^5)_\infty}. \tag{3.8.3}$$

Write each of the products on the right-hand side of (3.8.3) in the common base q^{15}. For example, $(q^3; q^3)_\infty = (q^3, q^6, q^9, q^{12}, q^{15}; q^{15})_\infty = (q^{3\pm}, q^{6\pm}, q^{15}; q^{15})_\infty$. Simplifying the result, we thus deduce that

$$\begin{aligned} &\frac{1}{(q^{1\pm}; q^5)_\infty^3 (q^{6\pm}; q^{15})_\infty} - \frac{1}{(q^{2\pm}; q^5)_\infty^3 (q^{3\pm}; q^{15})_\infty} \\ &\quad = \frac{3q}{(q^{1\pm}, q^{2\pm}, q^{3\pm}, q^{3\pm}, q^{4\pm}, q^{5\pm}, q^{5\pm}, q^{6\pm}, q^{6\pm}, q^{7\pm}; q^{15})_\infty}. \end{aligned} \tag{3.8.4}$$

Observe that the quotients on the left-hand side of (3.8.4) represent the generating functions for $p_1(n)$ and $p_2(n)$, and the right-hand side represents $3q$ times the generating function for $p_3(n)$. Hence, (3.8.4) is equivalent to

$$\sum_{n=0}^\infty p_1(n) q^n - \sum_{n=0}^\infty p_2(n) q^n = 3q \sum_{n=0}^\infty p_3(n) q^n. \tag{3.8.5}$$

The desired equality follows from equating coefficients on both sides of (3.8.5). □

From Theorem 3.8.2, we immediately obtain the following corollary.

Corollary 3.8.3. *Let $p_1(n)$ and $p_2(n)$ be as defined in Theorem 3.8.2. For each nonnegative integer n,*

$$p_1(n) \equiv p_2(n) \pmod 3. \tag{3.8.6}$$

It would be interesting to find a combinatorial explanation for (3.8.6).

Example 3.8.4. *We illustrate Theorem 3.8.2 in the case $n = 3$. Let the colors available be red (r), green (g), orange (o), and violet (v) when four are available, and red (r), green (g), and orange (o) when three are available. Then $p_1(3) = 10$, $p_2(3) = 4$, $p_3(2) = 2$, and the required partitions are:*

$$\begin{aligned}
p_1(3): \quad & 1_r + 1_r + 1_r = 1_g + 1_g + 1_g = 1_o + 1_o + 1_o = 1_r + 1_r + 1_g \\
& \quad = 1_r + 1_r + 1_o = 1_g + 1_g + 1_r = 1_g + 1_g + 1_o = 1_o + 1_o + 1_r \\
& \quad = 1_o + 1_o + 1_g = 1_r + 1_g + 1_o, \\
p_2(3): \quad & 3_r = 3_g = 3_o = 3_v, \\
p_3(2): \quad & 2 = 1 + 1.
\end{aligned}$$

Theorem 3.8.5. *Let $p_1(n)$ denote the number of partitions of n into parts congruent to $\pm 1 \pmod 5$ or $\pm 3, \pm 5 \pmod{15}$, where the parts congruent to $\pm 5 \pmod{15}$ have two colors and the parts congruent to $\pm 3 \pmod{15}$ have three colors.*

Let $p_2(n)$ denote the number of partitions of n into parts congruent to $\pm 2 \pmod 5$ or $\pm 5, \pm 6 \pmod{15}$, where the parts congruent to $\pm 5 \pmod{15}$ have two colors and the parts congruent to $\pm 6 \pmod{15}$ have three colors.

Let $p_3(n)$ denote the number of partitions of n into parts not divisible by 5, and with parts congruent to $\pm 3, \pm 6 \pmod{15}$ having two colors.

Define $p_i(0) := 1$ for $i = 1, 2, 3$, and $p_2(n) := 0$ for $n < 0$. Then, for each nonnegative integer n,

$$p_1(n) + p_2(n-2) = p_3(n).$$

Proof. Proceeding as in the proof of Theorem 3.8.2, we conclude that the identity in Theorem 3.3.1(ii) is equivalent to

$$\begin{aligned}
& \frac{1}{(q^{3\pm}; q^{15})_\infty^3 (q^{5\pm}; q^{15})_\infty^2 (q^{1\pm}; q^5)_\infty} + \frac{q^2}{(q^{6\pm}; q^{15})_\infty^3 (q^{5\pm}; q^{15})_\infty^2 (q^{2\pm}; q^5)_\infty} \\
& \quad = \frac{1}{(q^{1\pm}, q^{2\pm}, q^{4\pm}, q^{7\pm}; q^{15})_\infty (q^{3\pm}, q^{6\pm}; q^{15})_\infty^2}.
\end{aligned}$$

The desired equality follows by noting that the left-hand side of the last equation is the generating function for $p_1(n) + p_2(n-2)$, and the right-hand side is the generating function for $p_3(n)$. □

Example 3.8.6. *We verify Theorem 3.8.5 in the case $n = 6$. Let the colors available be red (r), green (g), and orange (o) when three are available, and red (r) and green (g) when two are available. Then $p_1(6) = 14$, $p_2(4) = 1$, $p_3(6) = 15$, and the required partitions are:*

$$\begin{aligned}
p_1(6): \quad & 6 = 5_r + 1 = 5_g + 1 = 4 + 1 + 1 = 3_r + 3_r = 3_g + 3_g = 3_o + 3_o \\
& \quad = 3_r + 3_g = 3_r + 3_o = 3_g + 3_o = 3_r + 1 + 1 + 1 = 3_g + 1 + 1 + 1 \\
& \quad = 3_o + 1 + 1 + 1 = 1 + 1 + 1 + 1 + 1 + 1, \\
p_2(4): \quad & 2 + 2, \\
p_3(6): \quad & 6_r = 6_g = 4 + 2 = 4 + 1 + 1 = 3_r + 3_r = 3_g + 3_g = 3_r + 3_g \\
& \quad = 3_r + 2 + 1 = 3_g + 2 + 1 = 3_r + 1 + 1 + 1 = 3_g + 1 + 1 + 1 \\
& \quad = 2 + 2 + 2 = 2 + 2 + 1 + 1 = 2 + 1 + 1 + 1 + 1 \\
& \quad = 1 + 1 + 1 + 1 + 1 + 1.
\end{aligned}$$

Next, we give applications of Theorem 3.5.1. Since Theorem 3.5.1 is an analogue of Theorem 3.3.1, the results in Theorems 3.8.7–3.8.10 are analogues of the results in Theorems 3.8.2–3.8.5.

Theorem 3.8.7. *Let $p_1(n)$ denote the number of partitions of n into parts where the odd parts are congruent to $\pm 1 \pmod 8$, with parts congruent to $\pm 9 \pmod{24}$ having four colors and the remaining odd parts having three colors; the even parts are congruent to $2 \pmod 4$ or $4 \pmod 8$, where the parts congruent to $4 \pmod 8$ have two colors.*

Let $p_2(n)$ denote the number of partitions of n into parts where the odd parts are congruent to $\pm 3 \pmod 8$, with the parts congruent to $\pm 3 \pmod{24}$ having four colors and the remaining odd parts having three colors; the even parts satisfy the same conditions as for $p_1(n)$.

Let $p_3(n)$ denote the number of partitions of n into parts where the even parts are congruent to $\pm 8 \pmod{24}$ and have two colors, and the odd parts also have two colors, except for parts congruent to $\pm 3, \pm 9 \pmod{24}$, which have three colors.

Define $p_i(0) := 1$ for $i = 1, 2, 3$, and $p_3(n) := 0$ for $n < 0$. Then, for each nonnegative integer n,

$$p_1(n) - p_2(n) = 3p_3(n-1).$$

Proof. Using (3.1.1), (3.1.2), (3.2.24), and Lemma 1.0.1, we write each of the functions appearing in Theorem 3.5.1(i) in terms of its product representation, and then simplify. We thus deduce that

$$\begin{aligned}
&\frac{1}{(q^{1\pm}, q^4; q^8)^3_\infty \, (q^{9\pm}, q^{12}; q^{24})_\infty} - \frac{1}{(q^{3\pm}, q^4; q^8)^3_\infty \, (q^{3\pm}, q^{12}; q^{24})_\infty} \\
&= 3q \frac{f^3(-q^{24})}{f^3(-q^8)} \cdot \frac{f(-q^2)f(-q^8)}{f(-q^4)} \cdot \frac{f^2(-q^4)}{f(-q^8)} \cdot \frac{f(-q^2)}{f(-q)f(-q^4)} \cdot \frac{f(-q^6)}{f(-q^3)f(-q^{12})} \\
&\quad \times \frac{f^2(-q^8)f(-q^2)}{f^2(-q^4)f(-q)}. \\
&= 3q \left(\frac{f(-q^8)}{f(-q^4)}\right)^3 \left(\frac{f(-q^4)}{f(-q^8)}\right)^2 \left(\frac{f(-q^2)}{f(-q)}\right)^2 \left(\frac{f(-q^6)}{f(-q^3)}\right) \left(\frac{f(-q^2)}{f(-q^4)}\right) \\
&\quad \times \left(\frac{f(-q^{24})}{f(-q^8)}\right)^2 \left(\frac{f(-q^{24})}{f(-q^{12})}\right) \\
&= 3q \frac{(q^2; q^4)_\infty (q^4; q^8)^2_\infty}{(q; q^2)^2_\infty (q^3; q^6)_\infty (q^4; q^8)^3_\infty (q^8, q^{16}; q^{24})^2_\infty (q^{12}; q^{24})_\infty}.
\end{aligned} \tag{3.8.7}$$

After a slight rearrangement and further simplification, we conclude from (3.8.7) that

$$\begin{aligned}
&\frac{1}{(q^{1\pm}; q^8)^3_\infty \, (q^{9\pm}; q^{24})_\infty \, (q^2; q^4)_\infty (q^4; q^8)^2_\infty} \\
&\quad - \frac{1}{(q^{3\pm}; q^8)^3_\infty \, (q^{3\pm}; q^{24})_\infty \, (q^2; q^4)_\infty (q^4; q^8)^2_\infty} \\
&\qquad = 3q \frac{1}{(q; q^2)^2_\infty (q^3; q^6)_\infty (q^{8\pm}; q^{24})^2_\infty}.
\end{aligned} \tag{3.8.8}$$

It is easy to see that (3.8.8) is equivalent to

$$\sum_{n=0}^{\infty} p_1(n)q^n - \sum_{n=0}^{\infty} p_2(n)q^n = 3q \sum_{n=0}^{\infty} p_3(n)q^n,$$

from which Theorem 3.8.7 readily follows. □

From Theorem 3.8.7, we immediately deduce the following corollary.

Corollary 3.8.8. *Let $p_1(n)$ and $p_2(n)$ be as defined in Theorem 3.8.7. For each nonnegative integer n,*

$$p_1(n) \equiv p_2(n) \pmod 3. \tag{3.8.9}$$

It would be interesting to have a direct combinatorial explanation of (3.8.9).

Example 3.8.9. *We illustrate Theorem 3.8.7 in the case $n = 3$. Let the colors available be red (r), green (g), orange (o), and violet (v). Then $p_1(3) = 13$, $p_2(3) = 4$, $p_3(2) = 3$, and the required partitions are:*

$$\begin{aligned}
p_1(3): \quad & 2 + 1_r = 2 + 1_g = 2 + 1_o = 1_r + 1_r + 1_r = 1_g + 1_g + 1_g \\
& \quad = 1_o + 1_o + 1_o = 1_r + 1_r + 1_g = 1_r + 1_r + 1_o = 1_g + 1_g + 1_r \\
& \quad = 1_g + 1_g + 1_o = 1_o + 1_o + 1_r = 1_o + 1_o + 1_g = 1_r + 1_g + 1_o, \\
p_2(3): \quad & 3_r = 3_g = 3_o = 3_v, \\
p_3(2): \quad & 1_r + 1_r = 1_g + 1_g = 1_r + 1_g.
\end{aligned}$$

The remaining theorems are also derived from modular identities for the Göllnitz–Gordon functions. The details of transforming the appropriate modular relations into identities involving partition functions are similar to the details given in the proof of Theorem 3.8.7; hence, we omit these details in the remaining proofs.

Theorem 3.8.10. *Let $p_1(n)$ denote the number of partitions of n into parts where odd parts are congruent to $\pm1, \pm3, \pm7 \pmod{24}$ with parts congruent to $\pm3 \pmod{24}$ having two colors, and even parts are congruent to $\pm6, \pm8, 12 \pmod{24}$ with parts congruent to $\pm8, 12 \pmod{24}$ having two colors.*

Let $p_2(n)$ denote the number of partitions of n into parts where odd parts are congruent to $\pm5, \pm9, \pm11 \pmod{24}$ with parts congruent to $\pm9 \pmod{24}$ having two colors, and even parts satisfy the same conditions as for $p_1(n)$.

Let $p_3(n)$ denote the number of partitions of n into parts not divisible by 2, and with parts congruent to $3 \pmod 6$ having two colors.

Let $\bar{p}_3(n)$ denote the number of partitions of n into distinct parts, with parts congruent to $3 \pmod 3$ having two colors.

Define $p_i(0) := \bar{p}_3(0) := 1$ for $i = 1, 2, 3$, and $p_2(n) := 0$ for $n < 0$. Then, for each nonnegative integer n,

$$p_1(n) + p_2(n-5) = p_3(n) = \bar{p}_3(n).$$

Proof. Set

$$P := \frac{1}{(q^{8\pm}, q^{12}; q^{24})_\infty^2 (q^{6\pm}; q^{24})_\infty}.$$

Then, the identity in Theorem 3.5.1(ii) is equivalent to

$$\frac{P}{(q^{3\pm};q^{24})_\infty^2(q^{1\pm},q^{7\pm};q^{24})_\infty}+q^5\frac{P}{(q^{9\pm};q^{24})_\infty^2(q^{5\pm},q^{11\pm};q^{24})_\infty}$$
$$=\frac{1}{(q;q^2)_\infty(q^3;q^6)_\infty}=(-q;q)_\infty(-q^3;q^3)_\infty, \tag{3.8.10}$$

where, to obtain the last equality, we applied (2.3.13). The required equalities follow from (3.8.10). □

Example 3.8.11. *We verify Theorem 3.8.10 in the case $n=6$. Let the colors available be red (r) and green (g). Then $p_1(6)=7$, $p_2(1)=0$, $p_3(6)=\bar{p}_3(6)=7$, and the required partitions are:*

$$\begin{aligned}
p_1(6):&\quad 6=3_r+3_r=3_g+3_g=3_r+3_g=3_r+1+1+1\\
&\quad\;\;=3_g+1+1+1=1+1+1+1+1+1,\\
p_2(1):&\quad \phi,\\
p_3(6):&\quad 5+1=3_r+3_r=3_g+3_g=3_r+3_g=3_r+1+1+1\\
&\quad\;\;=3_g+1+1+1=1+1+1+1+1+1,\\
\bar{p}_3(6):&\quad 6_r=6_g=5+1=4+2=3_r+3_g=3_r+2+1=3_g+2+1.
\end{aligned}$$

Next, we give partition-theoretic interpretations for the two identities in Theorem 3.6.1(i),(ii). For partition-theoretic interpretations of (iii) and (iv), see [58].

Theorem 3.8.12. *Let $p_1(n)$ denote the number of partitions of n where all the odd parts are distinct and are congruent to $\pm 1 \pmod 8$ with parts congruent to $\pm 9 \pmod{24}$ having two colors, and all the even parts are congruent to $4 \pmod 8$ or $6 \pmod{12}$ with parts congruent to $4 \pmod 8$ having two colors.*

Let $p_2(n)$ denote the number of partitions of n where all the odd parts are distinct and are congruent to $\pm 3 \pmod 8$ with parts congruent to $\pm 3 \pmod{24}$ having two colors, and all the even parts satisfy the same condition as for $p_1(n)$.

Define $p_i(0):=1$ for $i=1,2$, and $p_2(n):=0$ for $n<0$. Then, for any positive integer n,

$$p_1(n)=p_2(n-1).$$

Proof. Set

$$P:=(-q^{1\pm};q^8)_\infty(-q^{9\pm};q^{24})_\infty \qquad\text{and}\qquad Q:=(-q^{3\pm};q^8)_\infty(-q^{3\pm};q^{24})_\infty.$$

Then Theorem 3.6.1(i) is equivalent to

$$\frac{P}{(q^4;q^8)_\infty^2(q^6;q^{12})_\infty}-q\frac{Q}{(q^4;q^8)_\infty^2(q^6;q^{12})_\infty}=1,$$

where q has been replaced by $-q$. The desired result follows from the last equality. □

Example 3.8.13. *We verify Theorem 3.8.12 in the case $n=9$. Let the colors available be red (r) and green (g). Then $p_1(9)=5$, $p_2(8)=5$, and the required* *partitions are:*

$$\begin{aligned} p_1(9): &\qquad 9_r = 9_g = 4_r + 4_r + 1 = 4_g + 4_g + 1 = 4_r + 4_g + 1, \\ p_2(8): &\qquad 4_r + 4_r = 4_g + 4_g = 4_r + 4_g = 5 + 3_r = 5 + 3_g. \end{aligned}$$

Theorem 3.8.14. *Let $p_1(n)$ denote the number of partitions of n where odd parts are distinct and congruent to $3 \pmod 6$ or $\pm 5, \pm 11 \pmod{24}$, and even parts are congruent to $2 \pmod 4$ or $12 \pmod{24}$.*

Let $p_2(n)$ denote the number of partitions of n where odd parts are distinct and congruent to $3 \pmod 6$ or $\pm 1, \pm 7 \pmod{24}$, and even parts satisfy the same conditions as for $p_1(n)$.

Set $p_i(0) := 1$ for $i = 1, 2$, and $p_2(n) := 0$ for $n < 0$.

Then, for any positive integer n,

$$p_1(n) = \begin{cases} p_2(n-2), \\ \qquad \text{if} \quad 12 \nmid n, \\ p_2(n-2) + 2 \sum'_{0 \le j \le \sqrt{k}} (-1)^j p(k - j^2), \\ \qquad \text{if} \quad n = 12k, \end{cases}$$

where $\sum'$ indicates that the term corresponding to $j = 0$, i.e., $p(k)$, is weighted by $1/2$.

Proof. The identity in Theorem 3.6.1(ii) is equivalent to

$$\begin{aligned} &\frac{(-q^3;q^6)_\infty (-q^{5\pm}, -q^{11\pm}; q^{24})_\infty}{(q^2;q^4)_\infty (q^{12};q^{24})_\infty} - q^2 \frac{(-q^3;q^6)_\infty (-q^{1\pm}, -q^{7\pm}; q^{24})_\infty}{(q^2;q^4)_\infty (q^{12};q^{24})_\infty} \\ &= (q^{12};q^{24})_\infty = \frac{1}{(-q^{12};q^{12})_\infty}, \end{aligned} \tag{3.8.11}$$

where we have replaced q by $-q$ and used (2.3.13) to obtain the last equality. Hence, by (3.8.1),

$$\sum_{n=0}^\infty p_1(n) q^n - q^2 \sum_{n=0}^\infty p_2(n) q^n = 1 + \sum_{n=1}^\infty \left(p_e(n) - p_o(n)\right) q^{12n}. \tag{3.8.12}$$

Equating coefficients on both sides of (3.8.12), we deduce that

$$p_1(n) - p_2(n-2) = 0, \qquad \text{if} \quad 12 \nmid n, \tag{3.8.13}$$

and

$$p_1(n) - p_2(n-2) = p_e(k) - p_o(k), \qquad \text{if} \quad n = 12k. \tag{3.8.14}$$

Moreover, by the definition of $p_o(n)$ and by (3.8.2),

$$\begin{aligned} p_e(k) - p_o(k) &= -p(k) + 2 \sum_{0 \le j \le \sqrt{k}} (-1)^j p(k - j^2) \\ &= 2 \sum'_{0 \le j \le \sqrt{k}} (-1)^j p(k - j^2). \end{aligned} \tag{3.8.15}$$

Combining (3.8.13)–(3.8.15), we complete the proof. $\square$

Example 3.8.15. *We verify Theorem 3.8.14 in the case $n = 13$. Then,*

$p_1(13)=8$, $p_2(11)=8$, and the required partitions are:

$$\begin{aligned}
p_1(13): \qquad & 13 = 11+2 = 10+3 = 9+2+2 = 6+5+2 = 6+3+2+2\\
& \quad = 5+2+2+2+2 = 3+2+2+2+2+2,\\
p_2(11): \qquad & 10+1 = 9+2 = 7+3+1 = 7+2+2 = 6+3+2\\
& \quad = 6+2+2+1 = 3+2+2+2+2 = 2+2+2+2+2+1.
\end{aligned}$$

Each of the modular identities that we have considered can yield several theorems in the theory of partitions. We illustrate by recording two further consequences of Theorem 3.5.1(ii).

Theorem 3.8.16. *Let $p_1(n)$ denote the number of partitions of n where odd parts are distinct and congruent to $3 \pmod 6$ or $\pm 5, \pm 11 \pmod{24}$, and even parts have two colors and are congruent to $12 \pmod{24}$.*

Let $p_2(n)$ denote the number of partitions of n where odd parts are distinct and congruent to $3 \pmod 6$ or $\pm 1, \pm 7 \pmod{24}$, and even parts satisfy the same conditions as for $p_1(n)$.

Define $p_i(0) := 1$ for $i = 1, 2$, and $p_2(n) := 0$ for $n < 0$.

Then, for any nonnegative integer n,

$$p_1(n) = \begin{cases} p_2(n-2), \\ \qquad \text{if} \quad 2 \nmid n, \\ p_2(n-2) + 2 \sum\nolimits'_{0 \le j \le \sqrt{k}} (-1)^j p(k - j^2), \\ \qquad \text{if} \quad n = 2k, \end{cases}$$

where $\sum'$ has the same meaning as in Theorem 3.8.14.

Proof. A slight rearrangement of (3.8.11) yields

$$\begin{aligned}
&\frac{(-q^3;q^6)_\infty(-q^{5\pm},-q^{11\pm};q^{24})_\infty}{(q^{12};q^{24})^2_\infty} - q^2 \frac{(-q^3;q^6)_\infty(-q^{1\pm},-q^{7\pm};q^{24})_\infty}{(q^{12};q^{24})^2_\infty} \\
&= (q^2;q^4)_\infty = \frac{1}{(-q^2;q^2)_\infty}. \qquad (3.8.16)
\end{aligned}$$

Using (3.8.16) and proceeding as in the proof of Theorem 3.8.14, we complete the proof. □

Example 3.8.17. *We verify Theorem 3.8.16 in the case $n = 13$. Then, $p_1(13) = 1$, $p_2(11) = 1$, and the required partitions are:*

$$\begin{aligned}
p_1(13): \qquad & 13,\\
p_2(11): \qquad & 7+3+1.
\end{aligned}$$

Theorem 3.8.18. *Let $p_1(n)$ denote the number of partitions of n with odd parts congruent to $\pm 1, \pm 7 \pmod{24}$, and with even parts congruent to $12 \pmod{24}$ and having two colors.*

Let $p_2(n)$ denote the number of partitions of n with odd parts congruent to $\pm 5, \pm 11 \pmod{24}$, and with even parts satisfying the same conditions as for $p_1(n)$.

Let $\bar{p}(n)$ denote the number of partitions of n into distinct odd parts.

Define $p_i(0) := \bar{p}(0) := 1$ for $i = 1, 2$, and $p_2(n) := 0$ for $n < 0$. Then, for *any nonnegative integer n,*

$$p_1(n) - p_2(n-2) = \bar{p}(n).$$

Proof. We write Theorem 3.6.1(ii) in the equivalent form

$$\frac{1}{(q^{1\pm}, q^{7\pm}; q^{24})_\infty (q^{12}; q^{24})_\infty^2} - \frac{q^2}{(q^{5\pm}, q^{11\pm}; q^{24})_\infty (q^{12}; q^{24})_\infty^2} = (-q; q^2)_\infty, \tag{3.8.17}$$

whence the required equality follows. □

Example 3.8.19. *We verify Theorem 3.8.18 in the case $n = 13$. Let the colors available be red (r) and green (g). Then, $p_1(13) = 4$, $p_2(11) = 1$, $\bar{p}(13) = 3$, and the required partitions are:*

$$\begin{aligned}
p_1(13): \quad & 12_r + 1 = 12_g + 1 = 7+1+1+1+1+1+1 \\
& \quad = 1+1+1+1+1+1+1+1+1+1+1+1+1, \\
p_2(11): \quad & 11, \\
\bar{p}(13)\ : \quad & 13 = 9+3+1 = 7+5+1.
\end{aligned}$$

Chapter 4

Modular Equations Relating $R(q)$ and $R(q^5)$

4.1 Introduction

In this chapter, we prove one of Ramanujan's beautiful modular relations for $R(q)$, namely,

Theorem 4.1.1. *For $|q| < 1$, let*

$$u := R(q) := \frac{q^{1/5}}{1} \, \frac{q}{+\,1} \, \frac{q^2}{+\,1} \, \frac{q^3}{+\,1} + \cdots,$$

$$v := R(q^5) := \frac{q}{1} \, \frac{q^5}{+\,1} \, \frac{q^{10}}{+\,1} \, \frac{q^{15}}{+\,1} + \cdots.$$

Then,

$$u^5 = v\frac{1 - 2v + 4v^2 - 3v^3 + v^4}{1 + 3v + 4v^2 + 2v^3 + v^4}.$$

This elegant identity was communicated by Ramanujan [77, p. xxvii] in his first letter to Hardy, and was recorded on page 289 of his second notebook and page 365 of his Lost Notebook (see [16, pp. 18–19] and [7, p. 93]). The first proof of Theorem 4.1.1 is due to Rogers [82]. A second proof has been given by Watson [97], and a third proof has been found by K.G. Ramanathan [68]. Recently, Yi [102] offered a proof utilizing eta-function identities. Theorem 4.1.1 is connected to modular equations of degree five.

The proof that we offer is new, and gives a stronger theorem than proofs offered by previous authors. In particular, we give for the first time identities for the expressions that appear in the numerator and denominator of Ramanujan's identity. These identities are the content of Theorem 4.2.1. Furthermore, new relations involving fourth and fifth powers of the Rogers-Ramanujan functions are established. Lastly, we apply the identities in Theorem 4.2.1 to prove a related identity that has important connections with the theory of partitions, namely, Theorem 4.2.5.

4.2 Statement of Results

Before we state our results, we provide some notation that we employ throughout the remainder of this chapter.

Fix

$$\zeta := e^{2\pi i/5}.$$

Set

$$\alpha := \frac{1-\sqrt{5}}{2} \qquad \text{and} \qquad \beta := \frac{1+\sqrt{5}}{2}. \tag{4.2.1}$$

We frequently use the well-known facts that

$$\alpha = -\left(\zeta + \zeta^{-1}\right) \qquad \text{and} \qquad \beta = -\left(\zeta^2 + \zeta^3\right). \tag{4.2.2}$$

Define

$$A_{s,t} := A_{s,t}(q) := \prod_{i=s,\,t} \prod_{n=1}^{\infty} \frac{1}{1 + \alpha\left(\zeta^i q^{1/5}\right)^n + \left(\zeta^i q^{1/5}\right)^{2n}} \tag{4.2.3}$$

and

$$B_{s,t} := B_{s,t}(q) := \prod_{i=s,\,t} \prod_{n=1}^{\infty} \frac{1}{1 + \beta\left(\zeta^i q^{1/5}\right)^n + \left(\zeta^i q^{1/5}\right)^{2n}}. \tag{4.2.4}$$

We are now ready to state our results.

Theorem 4.2.1. *For $|q| < 1$, we have*

$$\begin{aligned}
\text{(i) } R^4(q) - 3R^3(q) + 4R^2(q) - 2R(q) + 1 &= R^2(q)\frac{1}{q^{2/5}}\frac{f^2(-q)}{f^2(-q^5)} \cdot A_{2,3} \cdot B_{1,4} \\
&= R^2(q)\frac{H^5(q^{1/5})}{H(q)}\left(\frac{f(-q^{1/5})}{q^{1/5}f(-q^5)}\right)^2, \\
\text{(ii)} R^4(q) + 2R^3(q) + 4R^2(q) + 3R(q) + 1 &= R^2(q)\frac{1}{q^{2/5}}\frac{f^2(-q)}{f^2(-q^5)} \cdot A_{1,4} \cdot B_{2,3} \\
&= R^2(q)\frac{G^5(q^{1/5})}{G(q)}\left(\frac{f(-q^{1/5})}{q^{1/5}f(-q^5)}\right)^2.
\end{aligned}$$

We note that Theorem 4.2.1 is new. Michael Somos [94], using PARI-GP code that he had written, empirically discovered the rightmost equalities in (i) and (ii) after the author shared with him the former equalities in (i) and (ii).

In Chapters 2 and 3, identities involving squares, cubes, and fourth powers of the Rogers–Ramanujan functions are studied, and applications to the Rogers–Ramanujan continued fraction are presented. The following corollaries provide new such relations involving fourth and fifth powers of the Rogers–Ramanujan functions.

Corollary 4.2.2. *We have*

$$\begin{aligned}
5\left[R(q^5) + \frac{1}{R(q^5)}\right] &= \left(\frac{f(-q^5)}{qf(-q^{25})}\right)^2 \left[A_{1,4}(q^5)B_{2,3}(q^5) - A_{2,3}(q^5)B_{1,4}(q^5)\right] \\
&= \left(\frac{f(-q)}{qf(-q^{25})}\right)^2 \left[\frac{G^5(q)}{G(q^5)} - \frac{H^5(q)}{H(q^5)}\right]
\end{aligned}$$

$$= \left(\frac{f(-q)}{qf(-q^{25})}\right)^2 \left(\frac{f(-q^5)}{f(-q^{25})}\right) \left[G^5(q)H(q^5) - H^5(q)G(q^5)\right].$$

Corollary 4.2.3. *We have*

$$\begin{aligned}
&\left[2R^2(q^5) - R(q^5) + 8 + \frac{1}{R(q^5)} + 2\frac{1}{R^2(q^5)}\right] \\
&= \left(\frac{f(-q^5)}{qf(-q^{25})}\right)^2 \left[A_{1,4}(q^5)B_{2,3}(q^5) + A_{2,3}(q^5)B_{1,4}(q^5)\right] \\
&= \left(\frac{f(-q)}{qf(-q^{25})}\right)^2 \left[\frac{G^5(q)}{G(q^5)} + \frac{H^5(q)}{H(q^5)}\right] \\
&= \left(\frac{f(-q)}{qf(-q^{25})}\right)^2 \left(\frac{f(-q^5)}{f(-q^{25})}\right) \left[G^5(q)H(q^5) + H^5(q)G(q^5)\right].
\end{aligned}$$

Corollary 4.2.4. *We have*

(i) $G^4(q^5) - 2qG^3(q^5)H(q^5) + 4q^2G^2(q^5)H^2(q^5) - 3q^3G(q^5)H^3(q^5) + q^4H^4(q^5)$

$$= A_{2,3}(q^5) \cdot B_{1,4}(q^5) = \frac{H^5(q)}{H(q^5)} \left(\frac{f(-q)}{f(-q^5)}\right)^2,$$

(ii) $G^4(q^5) + 3qG^3(q^5)H(q^5) + 4q^2G^2(q^5)H^2(q^5) + 2q^3G(q^5)H^3(q^5) + q^4H^4(q^5)$

$$= A_{1,4}(q^5) \cdot B_{2,3}(q^5) = \frac{G^5(q)}{G(q^5)} \left(\frac{f(-q)}{f(-q^5)}\right)^2.$$

Next, we record a theorem with important connections to the theory of partitions.

Theorem 4.2.5. *We have*

$$\begin{aligned}
\frac{f^6(-q^5)}{q^4 f(-q) f^5(-q^{25})} &= R^4(q^5) - R^3(q^5) + 2R^2(q^5) - 3R(q^5) + 5 \\
&\quad + 3R^{-1}(q^5) + 2R^{-2}(q^5) + R^{-3}(q^5) + R^{-4}(q^5). \qquad (4.2.5)
\end{aligned}$$

Theorem 4.2.5 is a well-known result of Ramanujan [72], [77, p. 212, identity (14)]. In [54], Hirschhorn gave a proof of an equivalent form of Theorem 4.2.5 and applied it to give an elegant proof of

$$\sum_{n\geq 0} p(5n+4)q^n = 5\frac{\left(q^5;q^5\right)_\infty^5}{(q;q)_\infty^6}, \qquad (4.2.6)$$

where $p(n)$ is the number of partitions of n, generated by $\sum_{n\geq 0} p(n)q^n = 1/(q)_\infty$. G.H. Hardy regarded (4.2.6) as Ramanujan's most beautiful identity [77, p. xxxv]; (4.2.6) also implies Ramanujan's famous congruence $p(5n+4) \equiv 0 \pmod 5$. In [55], Hirschhorn and Hunt utilized Theorem 4.2.5 to to give an elementary proof of Ramanujan's conjecture for $p(n)$ modulo arbitrary powers of five. Recently, Hirschhorn and Sellers [56] used Theorem 4.2.5 to prove congruence relations for broken k-diamond partitions, including one originally conjectured by Andrews and P. Paule [9].

4.3 Auxiliary Results

In order to prove Theorem 4.2.1, we dissect the products appearing in (4.2.3) and (4.2.4) according to the congruence class modulo 5 of the index. To describe this, we employ the following refinements of the reciprocals of (4.2.3) and (4.2.4): For $s,\, t,\, j \in \{1,\, 2,\, 3,\, 4,\, 5\}$, define

$$A_{s,t}^{j} := A_{s,t}^{j}(q) := \prod_{i=s,\,t} \prod_{\substack{n=1 \\ n\equiv j \pmod 5}}^{\infty} \left(1+\alpha\left(\zeta^i q\right)^n + \left(\zeta^i q\right)^{2n}\right) \tag{4.3.1}$$

and

$$B_{s,t}^{j} := B_{s,t}^{j}(q) := \prod_{i=s,\,t} \prod_{\substack{n=1 \\ n\equiv j \pmod 5}}^{\infty} \left(1+\beta\left(\zeta^i q\right)^n + \left(\zeta^i q\right)^{2n}\right). \tag{4.3.2}$$

Next, we record some technical lemmata for various products of (4.3.1) and (4.3.2).

Lemma 4.3.1. *We have*

(i) $$A_{1,4}^{1} \times B_{2,3}^{1} = \left(q;q^5\right)_\infty^3 \left(q^5;q^{25}\right)_\infty,$$

(ii) $$A_{1,4}^{4} \times B_{2,3}^{4} = \left(q^4;q^5\right)_\infty^3 \left(q^{20};q^{25}\right)_\infty,$$

(iii) $$A_{2,3}^{2} \times B_{1,4}^{2} = \left(q^2;q^5\right)_\infty^3 \left(q^{10};q^{25}\right)_\infty,$$

(iv) $$A_{2,3}^{3} \times B_{1,4}^{3} = \left(q^3;q^5\right)_\infty^3 \left(q^{15};q^{25}\right)_\infty.$$

Proof of (i), (ii). For $j=1$ and $j=4$,

$$\begin{aligned} A_{1,4}^{j} \times B_{2,3}^{j} &= \left\{\prod_{i=1,\,4} \prod_{\substack{n=1 \\ n\equiv j \pmod 5}}^{\infty} \left(1+\alpha\left(\zeta^i q\right)^n + \left(\zeta^i q\right)^{2n}\right)\right\} \\ &\qquad \times \left\{\prod_{i=2,\,3} \prod_{\substack{n=1 \\ n\equiv j \pmod 5}}^{\infty} \left(1+\beta\left(\zeta^i q\right)^n + \left(\zeta^i q\right)^{2n}\right)\right\} \\ &= \prod_{\substack{n=1 \\ n\equiv j \pmod 5}}^{\infty} \left\{\left(1+\alpha\left(\zeta q\right)^n + \left(\zeta q\right)^{2n}\right)\left(1+\alpha\left(\zeta^4 q\right)^n + \left(\zeta^4 q\right)^{2n}\right)\right. \\ &\qquad \left. \times \left(1+\beta\left(\zeta^2 q\right)^n + \left(\zeta^2 q\right)^{2n}\right)\left(1+\beta\left(\zeta^3 q\right)^n + \left(\zeta^3 q\right)^{2n}\right)\right\}. \end{aligned} \tag{4.3.3}$$

For each integer $n \geq 1$,

$$\begin{aligned} &\left(1+\alpha\left(\zeta q\right)^n + \left(\zeta q\right)^{2n}\right) \times \left(1+\alpha\left(\zeta^4 q\right)^n + \left(\zeta^4 q\right)^{2n}\right) \\ &\qquad = 1+\alpha q^n\left[\zeta^n+\zeta^{4n}\right] + q^{2n}\left[\zeta^{2n}+\alpha^2+\zeta^{8n}\right] + \alpha q^{3n}\left[\zeta^{6n}+\zeta^{9n}\right] + q^{4n}. \end{aligned} \tag{4.3.4}$$

Employing (4.2.2), we deduce that, for $n \equiv 1 \pmod 5$ and $n \equiv 4 \pmod 5$,

$$-\alpha = \zeta^n + \zeta^{4n} = \zeta^{6n} + \zeta^{9n} \tag{4.3.5}$$

and

$$-\beta = \zeta^{2n} + \zeta^{8n}. \tag{4.3.6}$$

Utilizing (4.3.5) and (4.3.6) in (4.3.4), we deduce that

$$\begin{aligned}\left(1 + \alpha\left(\zeta q\right)^n + \left(\zeta q\right)^{2n}\right) \times \left(1 + \alpha\left(\zeta^4 q\right)^n + \left(\zeta^4 q\right)^{2n}\right) \\ = 1 - \alpha^2 q^n + q^{2n}\left[\alpha^2 - \beta\right] - \alpha^2 q^{3n} + q^{4n}.\end{aligned} \tag{4.3.7}$$

Similarly, we deduce that, for $n \equiv 1 \pmod 5$ and $n \equiv 4 \pmod 5$,

$$\begin{aligned}&\left(1 + \beta\left(\zeta^2 q\right)^n + \left(\zeta^2 q\right)^{2n}\right) \times \left(1 + \beta\left(\zeta^3 q\right)^n + \left(\zeta^3 q\right)^{2n}\right) \\ &\quad = 1 + \beta q^n\left[\zeta^{2n} + \zeta^{3n}\right] + q^{2n}\left[\zeta^{4n} + \beta^2 + \zeta^{6n}\right] + \beta q^{3n}\left[\zeta^{7n} + \zeta^{8n}\right] + q^{4n} \\ &\quad = 1 - \beta^2 q^n + q^{2n}\left[\beta^2 - \alpha\right] - \beta^2 q^{3n} + q^{4n}.\end{aligned} \tag{4.3.8}$$

Multiply the right-hand side of (4.3.7) with the last expression in (4.3.8) and employ the relations $\alpha + \beta = 1$, $\alpha^2 + \beta^2 = 3$, $\alpha^3 + \beta^3 = 4$, and $\alpha\beta = -1$ to deduce that, for $n \equiv 1 \pmod 5$ and $n \equiv 4 \pmod 5$,

$$\begin{aligned}&\left(1 + \alpha\left(\zeta q\right)^n + \left(\zeta q\right)^{2n}\right)\left(1 + \alpha\left(\zeta^4 q\right)^n + \left(\zeta^4 q\right)^{2n}\right) \\ &\quad \times \left(1 + \beta\left(\zeta^2 q\right)^n + \left(\zeta^2 q\right)^{2n}\right)\left(1 + \beta\left(\zeta^3 q\right)^n + \left(\zeta^3 q\right)^{2n}\right) \\ &\quad = 1 - 3q^n + 3q^{2n} - q^{3n} - q^{5n} + 3q^{6n} - 3q^{7n} + q^{8n} \\ &\quad = \left(1 - q^n\right)^3\left(1 - q^{5n}\right).\end{aligned} \tag{4.3.9}$$

Substituting (4.3.9) into (4.3.3), we conclude that, for $j = 1$ and $j = 4$,

$$A_{1,4}^j \times B_{2,3}^j = \prod_{\substack{n=1 \\ n \equiv j \pmod 5}}^{\infty} \left(1 - q^n\right)^3\left(1 - q^{5n}\right). \tag{4.3.10}$$

For $j = 1$, the right-hand side of (4.3.10) equals

$$\left(q; q^5\right)_\infty^3 \left(q^5; q^{25}\right)_\infty,$$

from which we deduce (i). For $j = 4$, the right-hand side of (4.3.10) equals

$$\left(q^4; q^5\right)_\infty^3 \left(q^{20}; q^{25}\right)_\infty,$$

from which we deduce (ii). This concludes the proof of (i) and (ii). $\square$

Proof of (iii), (iv). Arguing as in the proof of (i) and (ii), we deduce that, for $j = 2$ and $j = 3$,

$$A_{2,3}^j \times B_{1,4}^j = \prod_{\substack{n=1 \\ n \equiv j \pmod 5}}^{\infty} \left(1 - q^n\right)^3\left(1 - q^{5n}\right). \tag{4.3.11}$$

Parts (iii) and (iv) follow from taking $j = 2$ and $j = 3$, respectively, in (4.3.11).

$\square$

Lemma 4.3.2. *The following are all equal to*

$$\frac{\left(q^{5j};q^{25}\right)_\infty}{\left(q^j;q^5\right)_\infty}:$$

(i) $A_{1,4}^j, \qquad j=2,\,3,$

(ii) $A_{2,3}^j, \qquad j=1,\,4,$

(iii) $B_{1,4}^j, \qquad j=1,\,4,$

(iv) $B_{2,3}^j, \qquad j=2,\,3.$

Proof. The proofs of all the claims in Lemma 4.3.2 are similar. Thus, we give details only for the $j=2$ case of (i). Now,

$$\begin{aligned}
A_{1,4}^2 &= \prod_{\substack{n=1\\ n\equiv 2 \pmod 5}}^{\infty} \left(1+\alpha\left(\zeta q\right)^n+\left(\zeta q\right)^{2n}\right)\left(1+\alpha\left(\zeta^4 q\right)^n+\left(\zeta^4 q\right)^{2n}\right)\\
&= \prod_{\substack{n=1\\ n\equiv 2 \pmod 5}}^{\infty} \Big(1+\alpha q^n\left[\zeta^n+\zeta^{4n}\right]+q^{2n}\left[\alpha^2+\zeta^{2n}+\zeta^{8n}\right]\\
&\qquad\qquad\qquad\qquad +\alpha q^{3n}\left[\zeta^{6n}+\zeta^{9n}\right]+q^{4n}\Big).
\end{aligned} \tag{4.3.12}$$

For $n\equiv 2 \pmod 5$, by (4.2.2), we find that

$$-\beta=\zeta^n+\zeta^{4n}=\zeta^{6n}+\zeta^{9n} \tag{4.3.13}$$

and

$$-\alpha=\zeta^{2n}+\zeta^{8n}. \tag{4.3.14}$$

Applying (4.3.13) and (4.3.14) and simplifying, we deduce from (4.3.12) that

$$\begin{aligned}
A_{1,4}^2 &= \prod_{\substack{n=1\\ n\equiv 2 \pmod 5}}^{\infty} \left(1-\alpha\beta q^n+q^{2n}\left[\alpha^2-\alpha\right]-\alpha\beta q^{3n}+q^{4n}\right)\\
&= \prod_{\substack{n=1\\ n\equiv 2 \pmod 5}}^{\infty} \left(1+q^n+q^{2n}+q^{3n}+q^{4n}\right)\\
&= \prod_{\substack{n=1\\ n\equiv 2 \pmod 5}}^{\infty} \frac{\left(1-q^{5n}\right)}{\left(1-q^n\right)}\\
&= \frac{\left(q^{10};q^{25}\right)_\infty}{\left(q^2;q^5\right)_\infty}.
\end{aligned} \tag{4.3.15}$$

This completes the proof. $\square$

Lemma 4.3.3. *We have*

$$\text{(i)} \qquad A_{1,4}^5 = A_{2,3}^5 = \prod_{n=1}^{\infty} \left(1 + 2\alpha q^{5n} + \left(2+\alpha^2\right) q^{10n} + 2\alpha q^{15n} + q^{20n}\right),$$

$$\text{(ii)} \qquad B_{1,4}^5 = B_{2,3}^5 = \prod_{n=1}^{\infty} \left(1 + 2\beta q^{5n} + \left(2+\beta^2\right) q^{10n} + 2\beta q^{15n} + q^{20n}\right),$$

$$\text{(iii)}\ A_{1,4}^5 \times B_{2,3}^5 = A_{2,3}^5 \times B_{1,4}^5 = \frac{\left(q^{25};q^{25}\right)_\infty^2}{\left(q^5;q^5\right)_\infty^2}.$$

Proof. Parts (i) and (ii) follow directly from definitions (4.3.1) and (4.3.2). To prove part (iii), first note that, by (i) and (ii), the first equality in (iii) is immediate. To prove the second equality in (iii), multiply the final expressions in (i) and (ii) and simplify with the help (4.2.1) to deduce that the leftmost expressions in (iii) are each equal to

$$\begin{aligned}
&\prod_{n=1}^{\infty} \left(1 + 2q^{5n} + 3q^{10n} + 4q^{15n} + 5q^{20n} + 4q^{25n} + 3q^{30n} + 2q^{35n} + q^{40n}\right) \\
&\quad = \prod_{n=1}^{\infty} \left(1 + q^{5n} + q^{10n} + q^{15n} + q^{20n}\right)^2 \\
&\quad = \prod_{n=1}^{\infty} \left(1 + q^{5n} + (q^{5n})^2 + (q^{5n})^3 + (q^{5n})^4\right)^2 \\
&\quad = \prod_{n=1}^{\infty} \left\{ \frac{\left(1-(q^{5n})^5\right)}{\left(1-q^{5n}\right)} \right\}^2 \\
&\quad = \frac{\left(q^{25};q^{25}\right)_\infty^2}{\left(q^5;q^5\right)_\infty^2}.
\end{aligned} \tag{4.3.16}$$

This completes the proof. □

We require one further remarkable set of identities. This result is recorded on page 206 of Ramanujan's Lost Notebook.

Theorem 4.3.4. [21, Theorem 4.1], [7, Entry 1.4.1, p. 21] *If α and β are defined by* (4.2.1)*, then*

$$\text{(i)} \qquad \frac{1}{\sqrt{R(q)}} - \alpha\sqrt{R(q)} = \frac{1}{q^{1/10}} \sqrt{\frac{f(-q)}{f(-q^5)}} \prod_{n=1}^{\infty} \frac{1}{1+\alpha q^{n/5} + q^{2n/5}},$$

$$\text{(ii)} \qquad \frac{1}{\sqrt{R(q)}} - \beta\sqrt{R(q)} = \frac{1}{q^{1/10}} \sqrt{\frac{f(-q)}{f(-q^5)}} \prod_{n=1}^{\infty} \frac{1}{1+\beta q^{n/5} + q^{2n/5}}.$$

In [68], Ramanathan offers a proof of these identities, but omits details for a key step. Berndt, Huang, Sohn, and Son give a complete proof in [21]. See also [7, pp. 21–24]. Recently, H.–C. Chan and S. Ebbing [33] have provided an intriguing, different approach to proving Theorem 4.3.4.

It has been observed that this theorem provides an amazing factorization of a fundamental modular identity of Ramanujan for the Rogers–Ramanujan continued fraction, namely,

$$\frac{1}{R(q)} - 1 - R(q) = \frac{f(-q^{1/5})}{q^{1/5} f(-q^5)}. \tag{4.3.17}$$

We show that much more is true; namely, we show that Theorems 4.2.1–4.2.5 and Theorem 4.1.1 can be derived starting from Theorem 4.3.4.

4.4 Proofs of Theorems 4.2.1–4.2.5 and Theorem 4.1.1

Proof of Theorem 4.2.1(i)*.* We require Theorem 4.3.4. Note that for each $i = 1, 2, 3, 4$, we obtain an identity from Theorem 4.3.4(i) by replacing $q^{1/5}$ with $\zeta^i q^{1/5}$. Observe that $R(q)$ is then replaced by $\zeta^i R(q)$. The same observations hold for Theorem 4.3.4(ii). We thus deduce that

$$\begin{aligned}
&\left\{\prod_{i=2,\,3}\left(\frac{1}{\sqrt{\zeta^i R(q)}} - \alpha\sqrt{\zeta^i R(q)}\right)\right\}\left\{\prod_{j=1,\,4}\left(\frac{1}{\sqrt{\zeta^j R(q)}} - \beta\sqrt{\zeta^j R(q)}\right)\right\} \\
&\quad = \left\{\prod_{i=2,\,3}\frac{1}{\sqrt{\zeta^i q^{1/5}}}\sqrt{\frac{f(-q)}{f(-q^5)}}\prod_{n=1}^{\infty}\frac{1}{1+\alpha\left(\zeta^i q^{1/5}\right)^n + \left(\zeta^i q^{1/5}\right)^{2n}}\right\} \\
&\quad\quad \times \left\{\prod_{j=1,\,4}\frac{1}{\sqrt{\zeta^j q^{1/5}}}\sqrt{\frac{f(-q)}{f(-q^5)}}\prod_{n=1}^{\infty}\frac{1}{1+\beta\left(\zeta^j q^{1/5}\right)^n + \left(\zeta^j q^{1/5}\right)^{2n}}\right\} \\
&\quad = \frac{1}{q^{2/5}}\frac{f^2(-q)}{f^2(-q^5)} \cdot A_{2,3} \cdot B_{1,4}.
\end{aligned} \tag{4.4.1}$$

Now, since $\alpha\beta = -1$, we deduce that

$$\begin{aligned}
\frac{1}{\sqrt{\zeta^i R(q)}} - \alpha\sqrt{\zeta^i R(q)} &= \frac{1 - \alpha\zeta^i R(q)}{\sqrt{\zeta^i R(q)}} \\
&= \frac{-\alpha\zeta^i\left(R(q) - \dfrac{1}{\alpha\zeta^i}\right)}{\sqrt{\zeta^i R(q)}} \\
&= \frac{-\alpha\zeta^{i/2}\left(R(q) + \beta\zeta^{-i}\right)}{\sqrt{R(q)}}.
\end{aligned} \tag{4.4.2}$$

Similarly, we find that

$$\frac{1}{\sqrt{\zeta^j R(q)}} - \beta\sqrt{\zeta^j R(q)} = \frac{-\beta\zeta^{j/2}\left(R(q) + \alpha\zeta^{-j}\right)}{\sqrt{R(q)}}. \tag{4.4.3}$$

Hence, utilizing (4.4.2) and (4.4.3), we deduce from (4.4.1) that

$$\begin{aligned}
&\frac{\left(R(q)+\alpha\zeta^{-1}\right)\left(R(q)+\alpha\zeta^{-4}\right)\left(R(q)+\beta\zeta^{-2}\right)\left(R(q)+\beta\zeta^{-3}\right)}{R^2(q)} \\
&\quad = \frac{1}{q^{2/5}}\frac{f^2(-q)}{f^2(-q^5)} \cdot A_{2,3} \cdot B_{1,4}.
\end{aligned} \tag{4.4.4}$$

Next, we expand the product on the left-hand side of (4.4.4) and simplify with the help of (4.2.1) and (4.2.2). We thus conclude that

$$\frac{R^4(q)-3R^3(q)+4R^2(q)-2R(q)+1}{R^2(q)}=\frac{1}{q^{2/5}}\frac{f^2(-q)}{f^2(-q^5)}\cdot A_{2,3}\cdot B_{1,4}, \tag{4.4.5}$$

from which the first equality of Theorem 4.2.1(i) follows.

In order to establish the second equality of Theorem 4.2.1(i), it suffices to prove that

$$\frac{f^2(-q^5)}{f^2(-q^{25})}\cdot A_{2,3}(q^5)\cdot B_{1,4}(q^5)=\frac{H^5(q)}{H(q^5)}\left(\frac{f(-q)}{f(-q^{25})}\right)^2, \tag{4.4.6}$$

where we have replaced q by q^5 in (i) and canceled the common factors of $R^2(q^5)/q^2$.

Recall the definitions (4.3.1) and (4.3.2) of $A^j_{s,t}$ and $B^j_{s,t}$, respectively. Employing (2.3.6) along with Lemmas 4.3.1–4.3.3, rearranging, and then applying (2.3.6), (1.0.22), and (1.0.23), we conclude that the left-hand side of (4.4.6) is equal to

$$\begin{aligned}
&\left(\frac{f(-q)}{f(-q^{25})}\right)^2\left(\frac{f(-q^5)}{f(-q)}\right)^2\left(\frac{1}{\left(\prod_{j=1}^5 A^j_{2,3}\right)\left(\prod_{j=1}^5 B^j_{1,4}\right)}\right)\\
&=\left(\frac{f(-q)}{f(-q^{25})}\right)^2(G(q)H(q))^2\\
&\quad\times\left(\frac{1}{\left(A^2_{2,3}\times B^2_{1,4}\right)\left(A^3_{2,3}\times B^3_{1,4}\right)\left(A^1_{2,3}\right)\left(A^4_{2,3}\right)\left(B^1_{1,4}\right)\left(B^4_{1,4}\right)\left(A^5_{2,3}\times B^5_{1,4}\right)}\right)\\
&=\left(\frac{f(-q)}{f(-q^{25})}\right)^2(G(q)H(q))^2\cdot\left(\frac{1}{(q^2;q^5)_\infty(q^3;q^5)_\infty}\right)^3\\
&\quad\cdot\left(\frac{1}{(q^{10};q^{25})_\infty(q^{15};q^{25})_\infty}\right)\cdot\left(\frac{(q;q^5)_\infty(q^4;q^5)_\infty}{(q^5;q^{25})_\infty(q^{20};q^{25})_\infty}\right)^2\cdot\left(\frac{(q^5;q^5)_\infty}{(q^{25};q^{25})_\infty}\right)^2\\
&=\left(\frac{f(-q)}{f(-q^{25})}\right)^2(G(q)H(q))^2\left(H^3(q)H(q^5)\right)\left(\frac{G^2(q^5)}{G^2(q)}\right)\left(\frac{1}{G^2(q^5)H^2(q^5)}\right)\\
&=\left(\frac{f(-q)}{f(-q^{25})}\right)^2\frac{H^5(q)}{H(q^5)}.
\end{aligned} \tag{4.4.7}$$

This establishes (4.4.6), and hence completes the proof of (i). □

Proof of Theorem 4.2.1(ii). The proof of (ii) is similar to the proof of (i). We therefore only sketch the details.

Proceeding as in the proof of (i), we deduce from Theorem 4.3.4 that

$$\begin{aligned}
&\left\{\prod_{i=1,\,4}\left(\frac{1}{\sqrt{\zeta^i R(q)}}-\alpha\sqrt{\zeta^i R(q)}\right)\right\}\left\{\prod_{j=2,\,3}\left(\frac{1}{\sqrt{\zeta^j R(q)}}-\beta\sqrt{\zeta^j R(q)}\right)\right\}\\
&\qquad=\frac{1}{q^{2/5}}\frac{f^2(-q)}{f^2(-q^5)}\cdot A_{1,4}\cdot B_{2,3}.
\end{aligned} \tag{4.4.8}$$

Simplifying as in the proof of (i), we deduce (ii). □

Next, we prove the corollaries of Theorem 4.2.1.

Proof of Corollary 4.2.2. Subtracting (i) from (ii) in Theorem 4.2.1, dividing by $R^2(q)$, and replacing q by q^5, we deduce immediately the first two equalities in Corollary 4.2.2. We obtain the third equality from the second by adding the fractions on the right-hand side of the second equality and simplifying with the help of (2.3.6). $\square$

Proof of Corollary 4.2.3. Add (i) and (ii) in Theorem 4.2.1 and proceed as in the proof of Corollary 4.2.2. $\square$

Proof of Corollary 4.2.4. To prove the first equality of (i), replace q by q^5 in the first equality of Theorem 4.2.1(i), and use (2.1.4) to write the resulting equation in the form

$$\begin{aligned}\frac{q^4H^4(q^5)}{G^4(q^5)} - 3\frac{q^3H^3(q^5)}{G^3(q^5)} + 4\frac{q^2H^2(q^5)}{G^2(q^5)} - 2\frac{qH(q^5)}{G(q^5)} + 1 \\ = \frac{q^2H^2(q^5)}{G^2(q^5)}\frac{1}{q^2}\frac{f^2(-q^5)}{f^2(-q^{25})} \cdot A_{2,3}(q^5) \cdot B_{1,4}(q^5).\end{aligned} \tag{4.4.9}$$

Multiplying (4.4.9) by $G^4(q^5)$ and simplifying the right-hand side of the resulting equation with the help of (2.3.6), we deduce the first equality of Corollary 4.2.4(i). The proof of the second equality in (i) is similar and utilizes the second equality in Theorem 4.2.1(i). The proof of (ii) proceeds from Theorem 4.2.1(ii) in the same manner as the proof of (i). We omit the details. $\square$

Next, we apply Theorem 4.2.1 to give rapid proofs of the elegant identities in Theorems 4.2.5 and 4.1.1.

Proof of Theorem 4.2.5. Replace q by q^5 in Theorem 4.2.1(i) and (ii), and multiply together the resulting expressions, employing the second equality in each case. We thus deduce that

$$\begin{aligned}&\left(R^4(q^5) - 3R^3(q^5) + 4R^2(q^5) - 2R(q^5) + 1\right) \\ &\times \left(R^4(q^5) + 2R^3(q^5) + 4R^2(q^5) + 3R(q^5) + 1\right) \\ &= R^4(q^5)\frac{(G(q)H(q))^5}{G(q^5)H(q^5)}\left(\frac{f(-q)}{qf(-q^{25})}\right)^4.\end{aligned} \tag{4.4.10}$$

Applying (2.3.6) six times, we find that

$$\begin{aligned}\frac{(G(q)H(q))^5}{G(q^5)H(q^5)}\left(\frac{f(-q)}{qf(-q^{25})}\right)^4 &= \left(\frac{f^5(-q^5)}{f^5(-q)}\right)\left(\frac{f(-q^5)}{f(-q^{25})}\right)\left(\frac{f^4(-q)}{q^4f^4(-q^{25})}\right) \\ &= \frac{f^6(-q^5)}{q^4f(-q)f^5(-q^{25})}.\end{aligned} \tag{4.4.11}$$

Employing (4.4.11) in (4.4.10), dividing by $R^4(q^5)$, and simplifying, we easily deduce Theorem 4.2.5. $\square$

Proof of Theorem 4.1.1. Recall from the statement of Theorem 4.1.1 the notation $u = R(q)$ and $v = R(q^5)$.

In Theorem 4.2.1(i) and (ii), replace q by q^5. Using the second equality in each case, divide (i) by (ii), and employ the notation of Theorem 4.1.1. We consequently find that

$$\frac{v^4 - 3v^3 + 4v^2 - 2v + 1}{v^4 + 2v^3 + 4v^2 + 3v + 1} = \left(\frac{q^{1/5}H(q)}{G(q)}\right)^5 \left(\frac{G(q^5)}{qH(q^5)}\right). \tag{4.4.12}$$

Employing (2.1.4), we see that

$$\left(\frac{q^{1/5}H(q)}{G(q)}\right)^5 \left(\frac{G(q^5)}{qH(q^5)}\right) = \frac{R^5(q)}{R(q^5)} = \frac{u^5}{v}. \tag{4.4.13}$$

Utilizing (4.4.13) in (4.4.12) and rearranging, we complete the proof of Theorem 4.1.1. □

Chapter 5

Further Identities for the Rogers–Ramanujan Functions

5.1 Introduction

In recent years, there has been growing interest and progress in finding new relations for the Rogers–Ramanujan functions that are in the spirit of Ramanujan's 40 identities for the Rogers–Ramanujan functions. S. Robins [80] discovered four new relations using the theory of modular forms. Three of these relations have been proved in this thesis ((2.1.5), (2.1.6), Theorem 3.3.1(i)). B. Gordon and R. McIntosh [46] rediscovered (2.1.5) and (2.1.6) in their work on transformation formulas for mock theta functions. As a consequence of his work on modular relations for the Göllnitz–Gordon functions, S.–S. Huang [58] discovered another new relation for the Rogers–Ramanujan functions. Applying an idea of G.N. Watson, B.C. Berndt and H. Yesilyurt [23] generated and proved a large number of new identities. Further new identities for the Rogers–Ramanujan functions are offered by the author in this thesis.

Recently, M. Koike [63] and M. Somos [93] have used computers to search for new relations. Koike's work was significantly inspired by connections between these types of identities and Thompson series in the theory of modular forms. See [63] for more details. Using the theory of modular forms, the identities conjectured by Koike have now been proved by K. Bringmann and H. Swisher [29, 30].

Some of the Koike identities are found on Ramanujan's list of 40 identities for the Rogers–Ramanujan functions, and have been proved using elementary methods. Others have been proved only by the theory of modular forms. The purpose of this chapter is to give new, elementary proofs in the spirit of Ramanujan of several of the identities of Koike and Somos.

A variety of methods are employed, including using known identities for the Rogers–Ramanujan functions, even-odd dissections of the Rogers-Ramanujan functions, the theory of modular equations, and a general theorem for expressing a product of two theta functions as a certain sum of products of pairs of theta functions, namely Theorem 1.0.3.

5.2 Preliminary Results

In the sections that follow, we require Watson's lemma [99] giving the even-odd dissections of the Rogers–Ramanujan functions. Watson's proof is based on Theorem 2.5.1(i) and (ii), for which we provided new proofs in Section 2.5. For an entirely different proof of Watson's lemma, see [19, pp. 19–20].

Lemma 5.2.1. *We have*

$$G(q) = \frac{f(-q^8)}{f(-q^2)}\left(G(q^{16}) + qH(-q^4)\right), \tag{5.2.1}$$

$$H(q) = \frac{f(-q^8)}{f(-q^2)}\left(q^3H(q^{16}) + G(-q^4)\right). \tag{5.2.2}$$

In the work that follows, we make use of several known identities for the Rogers–Ramanujan functions. For convenience, we list some of the required results here.

From [19, pp. 8–9], we require Entries 3.6–3.8, 3.11, 3.12, 3.15, and 3.16.

$$G(q)G(q^9) + q^2H(q)H(q^9) = \frac{f^2(-q^3)}{f(-q)f(-q^9)}, \tag{5.2.3}$$

$$G(q^2)G(q^3) + qH(q^2)H(q^3) = \frac{\chi(-q^3)}{\chi(-q)}, \tag{5.2.4}$$

$$G(q^6)H(q) - qG(q)H(q^6) = \frac{\chi(-q)}{\chi(-q^3)}, \tag{5.2.5}$$

$$G(q^8)H(q^3) - qG(q^3)H(q^8) = \frac{\chi(-q)\chi(-q^4)}{\chi(-q^3)\chi(-q^{12})}, \tag{5.2.6}$$

$$G(q)G(q^{24}) + q^5H(q)H(q^{24}) = \frac{\chi(-q^3)\chi(-q^{12})}{\chi(-q)\chi(-q^4)}, \tag{5.2.7}$$

$$G(q^3)G(q^7) + q^2H(q^3)H(q^7) = G(q^{21})H(q) - q^4G(q)H(q^{21}) \tag{5.2.8}$$

$$= \frac{1}{2\sqrt{q}}\chi(q^{1/2})\chi(-q^{3/2})\chi(q^{7/2})\chi(-q^{21/2}) - \frac{1}{2\sqrt{q}}\chi(-q^{1/2})\chi(q^{3/2})\chi(-q^{7/2})\chi(q^{21/2}), \tag{5.2.9}$$

$$G(q^2)G(q^{13}) + q^3H(q^2)H(q^{13}) = G(q^{26})H(q) - q^5G(q)H(q^{26}) \tag{5.2.10}$$

$$= \sqrt{\frac{\chi(-q^{13})}{\chi(-q)} - q\frac{\chi(-q)}{\chi(-q^{13})}}. \tag{5.2.11}$$

We further apply some results from [23].

Theorem 5.2.2. [23, Equations 5.4, 5.6; 4.31, 4.36, 4.28, 4.39]

$$G(q^{14})G(q^{16}) + q^6 H(q^{14})H(q^{16}) = \frac{1}{2}\chi(-q^4)\left\{\chi(q)\chi(-q^7) + \chi(-q)\chi(q^7)\right\}, \tag{5.2.12}$$

$$G(q^{112})H(q^2) - q^{22}H(q^{112})G(q^2) = \frac{1}{2}\chi(-q^{28})\left\{\chi(q)\chi(-q^7) + \chi(-q)\chi(q^7)\right\}, \tag{5.2.13}$$

$$\begin{aligned} &G(q^{42})H(-q^2) + q^8 H(q^{42})G(-q^2) \\ &\qquad = \frac{1}{2q}\frac{\chi(-q^{12})}{\chi(q^2)}\left\{\chi(q)\chi(-q^{21}) - \chi(-q)\chi(q^{21})\right\}, \end{aligned} \tag{5.2.14}$$

$$\begin{aligned} &G(q^6)G(-q^{14}) - q^4 H(q^6)H(-q^{14}) \\ &\qquad = \frac{1}{2q^3}\frac{\chi(-q^{84})}{\chi(q^{14})}\left\{\chi(q^3)\chi(-q^7) - \chi(-q^3)\chi(q^7)\right\} \end{aligned} \tag{5.2.15}$$

$$G(q^8)G(q^{42}) + q^{10}H(q^8)H(q^{42}) = \frac{1}{2}\frac{\chi(-q^4)}{\chi(q^6)}\left\{\chi(q)\chi(-q^{21}) + \chi(-q)\chi(q^{21})\right\}, \tag{5.2.16}$$

$$G(q^{56})H(q^6) - q^{10}H(q^{56})G(q^6) = \frac{1}{2}\frac{\chi(-q^{28})}{\chi(q^{42})}\left\{\chi(q^3)\chi(-q^7) + \chi(-q^3)\chi(q^7)\right\}. \tag{5.2.17}$$

Theorem 5.2.3. [23, Theorem 1.1] *Let*

$$\begin{aligned} B(q) &:= G(q^{12})H(-q^7) + qG(-q^7)H(q^{12}), \\ C(q) &:= G(q)G(q^{84}) + q^{17}H(q)H(q^{84}), \\ V(q) &:= H(-q)G(q^{21}) + q^4 G(-q)H(q^{21}), \\ W(q) &:= G(q^4)G(q^{21}) + q^5 H(q^4)H(q^{21}), \\ Z(q) &:= H(q^3)G(q^{28}) - q^5 G(q^3)H(q^{28}), \\ Y(q) &:= G(q^3)G(-q^7) - q^2 H(q^3)H(-q^7). \end{aligned}$$

Then,

$$\frac{C(q^2)}{Y(-q^2)} = \frac{V(-q^2)}{B(-q^2)} = \frac{C(q)}{B(q)} = \frac{f(-q^{12})f(-q^{14})}{f(-q^2)f(-q^{84})} \tag{5.2.18}$$

and

$$\frac{Z(-q)}{W(q)} = \frac{Z(q)}{W(-q)} = \frac{Y(q^2)}{W(q^2)} = \frac{Z(q^2)}{V(q^2)} = \frac{f(-q^4)f(-q^{42})}{f(-q^6)f(-q^{28})}. \tag{5.2.19}$$

Huang derived an identity that belongs to the same class of identities as those in Theorem 5.2.3, but is different from any identity in (5.2.18) or (5.2.19). In particular, Huang proved the following theorem.

Theorem 5.2.4. [58, Equation 2.28] *With $W(q)$ and $C(q)$ defined as in the* *preceding theorem, we have*

$$W(q)C(q)=\frac{1}{2q}\left\{\frac{f^3(-q^2)f^2(-q^3)f(-q^{12})f(-q^{14})}{f^2(-q)f^2(-q^4)f^2(-q^6)f(-q^{84})}\right.$$
$$\left.-\frac{f(-q^6)f^2(-q^7)f(-q^{28})f^3(-q^{42})}{f(-q^4)f^2(-q^{14})f^2(-q^{21})f^2(-q^{84})}\right\}. \tag{5.2.20}$$

Lastly, we make a definition. Identities for the Rogers–Ramanujan functions usually involve the Rogers–Ramanujan functions in the combinations

$$G(q^a)G(q^b)+q^{(a+b)/5}H(q^a)H(q^b)$$

or

$$G(q^a)H(q^b)-q^{(a-b)/5}G(q^b)H(q^a).$$

Put

$$N=ab.$$

Following Koike [63], we call N the level of the identity.

5.3 Identities Connecting the Rogers–Ramanujan Functions at the Arguments q and $-q$

Identity (5.3.1) below was discovered empirically by Somos [94], and communicated to the author. We offer, to our knowledge, the first proof of this result. The second identity (5.3.2) is a natural companion found by the author.

Theorem 5.3.1. *We have*

$$\begin{aligned}&G(-q^2)H(-q^2)\left[G(q)H(q)-G(-q)H(-q)\right]\\&=G(q^2)H(q^2)\left[G(q)H(-q)-G(-q)H(q)\right]\end{aligned} \tag{5.3.1}$$

and

$$\begin{aligned}&G(q^4)H(q^4)\left[G(q)H(q)+G(-q)H(-q)\right]\\&=G(q^2)G(-q^2)H(q^2)H(-q^2)\left[G(q)H(-q)+G(-q)H(q)\right].\end{aligned} \tag{5.3.2}$$

Proof of (5.3.1). From Theorem 2.5.1(i) and (ii), we easily deduce that

$$G(q)=\frac{\phi(q)+\phi(q^5)}{2G(q^4)f(-q^2)}\qquad\text{and}\qquad H(q)=\frac{\phi(q)-\phi(q^5)}{2qH(q^4)f(-q^2)}. \tag{5.3.3}$$

By Entry 10(iv) in Chapter 19 of Ramanujan's second notebook [76], [14, p. 262], we know that

$$\phi^2(q)-\phi^2(q^5)=4qf(q,q^9)f(q^3,q^7). \tag{5.3.4}$$

By (1.0.17) with $a = q$, $b = q^9$, $c = q^3$, and $d = q^7$, we find that

$$f(q, q^9)f(q^3, q^7) - f(-q, -q^9)f(-q^3, -q^7) = 2qf(q^6, q^{14})f(q^2, q^{18}). \tag{5.3.5}$$

Applying (5.3.3), (5.3.4), and (5.3.5), we deduce that

$$\begin{aligned}
&G(q)H(q) - G(-q)H(-q) \\
&= \frac{\big(\phi(q) + \phi(q^5)\big)\big(\phi(q) - \phi(q^5)\big)}{4qG(q^4)H(q^4)f^2(-q^2)} + \frac{\big(\phi(-q) + \phi(-q^5)\big)\big(\phi(-q) - \phi(-q^5)\big)}{4qG(q^4)H(q^4)f^2(-q^2)} \\
&= \frac{1}{4qG(q^4)H(q^4)f^2(-q^2)}\left[\big(\phi^2(q) - \phi^2(q^5)\big) + \big(\phi^2(-q) - \phi^2(-q^5)\big)\right] \\
&= \frac{4q}{4qG(q^4)H(q^4)f^2(-q^2)}\left[f(q, q^9)f(q^3, q^7) - f(-q, -q^9)f(-q^3, -q^7)\right] \\
&= \frac{2q}{G(q^4)H(q^4)f^2(-q^2)}f(q^6, q^{14})f(q^2, q^{18}). \qquad (5.3.6)
\end{aligned}$$

Similarly, by (5.3.3), [14, Entry 25(iii), p. 40], namely,

$$\phi(q)\phi(-q) = \phi^2(-q^2), \tag{5.3.7}$$

and by (5.3.4), we find that

$$\begin{aligned}
&G(q)H(-q) - G(-q)H(q) \\
&= \frac{\big(\phi(q) + \phi(q^5)\big)\big(\phi(-q) - \phi(-q^5)\big)}{-4qG(q^4)H(q^4)f^2(-q^2)} - \frac{\big(\phi(-q) + \phi(-q^5)\big)\big(\phi(q) - \phi(q^5)\big)}{4qG(q^4)H(q^4)f^2(-q^2)} \qquad (5.3.8) \\
&= \frac{-2}{4qG(q^4)H(q^4)f^2(-q^2)}\left[\phi(q)\phi(-q) - \phi(q^5)\phi(-q^5)\right] \\
&= \frac{-2}{4qG(q^4)H(q^4)f^2(-q^2)}\left[\phi^2(-q^2) - \phi^2(-q^{10})\right] \\
&= \frac{2q}{G(q^4)H(q^4)f^2(-q^2)}f(-q^2, -q^{18})f(-q^6, -q^{14}). \qquad (5.3.9)
\end{aligned}$$

Employing (5.3.6) and (5.3.9) in (5.3.1), replacing q^2 by q, and simplifying, we deduce that (5.3.1) is equivalent to

$$\frac{G(-q)H(-q)}{G(q)H(q)} = \frac{f(-q, -q^9)f(-q^3, -q^7)}{f(q, q^9)f(q^3, q^7)}. \tag{5.3.10}$$

We prove (5.3.10). By (2.3.6),

$$\frac{G(-q)H(-q)}{G(q)H(q)} = \frac{f(-q)f(q^5)}{f(-q^5)f(q)}. \tag{5.3.11}$$

By the Jacobi Triple Product Identity (1.0.6) and by (1.0.9),

$$\begin{aligned}
f(-q, -q^9)f(-q^3, -q^7) &= (q; q^{10})_\infty(q^3; q^{10})_\infty(q^7; q^{10})_\infty(q^9; q^{10})_\infty(q^{10}; q^{10})^2_\infty \\
&= \frac{(q; q^2)_\infty(q^{10}; q^{10})^2_\infty}{(q^5; q^{10})_\infty} = \frac{(q; q)_\infty(q^{10}; q^{10})^3_\infty}{(q^2; q^2)_\infty(q^5; q^5)_\infty} \\
&= \frac{f(-q)f^3(-q^{10})}{f(-q^2)f(-q^5)}. \qquad (5.3.12)
\end{aligned}$$

Replacing q by $-q$ in (5.3.12), we see that

$$f(q,q^9)f(q^3,q^7)=\frac{f(q)f^3(-q^{10})}{f(-q^2)f(q^5)}. \tag{5.3.13}$$

Combining (5.3.12) and (5.3.13), we arrive at

$$\frac{f(-q,-q^9)f(-q^3,-q^7)}{f(q,q^9)f(q^3,q^7)}=\frac{f(-q)f(q^5)}{f(-q^5)f(q)}. \tag{5.3.14}$$

Comparing (5.3.11) with (5.3.14), we easily deduce the truth of (5.3.10). This completes the proof of (5.3.1). □

Proof of (5.3.2). The proof is similar to the proof of (5.3.1). By (1.0.16) with $a=q$, $b=q^9$, $c=q^3$, and $d=q^7$,

$$f(q,q^9)f(q^3,q^7)+f(-q,-q^9)f(-q^3,-q^7)=2f(q^4,q^{16})f(q^8,q^{12}). \tag{5.3.15}$$

Apply (5.3.3), (5.3.4), and (5.3.15), and proceed as in the proof of (5.3.1) to deduce that

$$G(q)H(q)+G(-q)H(-q)=\frac{2}{G(q^4)H(q^4)f^2(-q^2)}f(q^4,q^{16})f(q^8,q^{12}). \tag{5.3.16}$$

Similarly, apply (5.3.3) and (5.3.4) to discover that

$$\begin{aligned}&G(q)H(-q)+G(-q)H(q)\\&=\frac{1}{2qG(q^4)H(q^4)f^2(-q^2)}\left[\phi(q)\phi(-q^5)-\phi(-q)\phi(q^5)\right].\end{aligned} \tag{5.3.17}$$

By (1.0.7),

$$\phi(q)\phi(-q^5)-\phi(-q)\phi(q^5)=f(q,q)f(-q^5,-q^5)-f(-q,-q)f(q^5,q^5). \tag{5.3.18}$$

To simplify the expression in (5.3.18), we first apply Theorem 1.0.3 with $a=b=q$, $c=d=q^5$, $\alpha=1$, $\beta=5$, $\epsilon_1=0$, $\epsilon_2=1$, and $m=6$. Using also (1.0.4) and (1.0.5), we find that

$$\begin{aligned}&f(q,q)f(-q^5,-q^5)\\&\quad=f(-q^6,-q^6)f(-q^{30},-q^{30})-q^5f(-q^{-4},-q^{16})f(-q^{20},-q^{40})\\&\qquad+q^{20}f(-q^{-14},-q^{26})f(-q^{10},-q^{50})-q^{45}f(-q^{-24},-q^{36})f(-1,-q^{60})\\&\qquad+q^{80}f(-q^{-34},-q^{46})f(-q^{-10},-q^{70})-q^{125}f(-q^{-44},-q^{56})f(-q^{-20},-q^{80})\\&\quad=f(-q^6,-q^6)f(-q^{30},-q^{30})+2qf(-q^4,-q^8)f(-q^{20},-q^{40})\\&\qquad+2q^4f(-q^2,-q^{10})f(-q^{10},-q^{50}).\end{aligned} \tag{5.3.19}$$

Replace q by $-q$ in (5.3.19) and subtract the result from (5.3.19) to arrive at

$$f(q,q)f(-q^5,-q^5)-f(-q,-q)f(q^5,q^5)=4qf(-q^4,-q^8)f(-q^{20},-q^{40}). \tag{5.3.20}$$

Applying the Jacobi Triple Product Identity (1.0.6) and writing the resulting products in the common base q^{60}, it is easy to show that

$$f(-q^4,-q^8)f(-q^{20},-q^{40})=f(-q^4,-q^{16})f(-q^8,-q^{12}). \tag{5.3.21}$$

Hence,

$$f(q,q)f(-q^5,-q^5) - f(-q,-q)f(q^5,q^5) = 4qf(-q^4,-q^{16})f(-q^8,-q^{12}). \tag{5.3.22}$$

Employ (5.3.18) and (5.3.22) to rewrite (5.3.17) in the form

$$G(q)H(-q) + G(q)H(-q) = \frac{2f(-q^4,-q^{16})f(-q^8,-q^{12})}{G(q^4)H(q^4)f^2(-q^2)}. \tag{5.3.23}$$

By (5.3.16) and (5.3.23), we see that (5.3.2) is equivalent to

$$\begin{aligned}&G(q^4)H(q^4)\left[\frac{2f(q^4,q^{16})f(q^8,q^{12})}{G(q^4)H(q^4)f^2(-q^2)}\right]\\ &= G(q^2)G(-q^2)H(q^2)H(-q^2)\left[\frac{2f(-q^4,-q^{16})f(-q^8,-q^{12})}{G(q^4)H(q^4)f^2(-q^2)}\right],\end{aligned} \tag{5.3.24}$$

or, after replacing q^2 by q and simplifying,

$$\frac{f(q^2,q^8)f(q^4,q^6)}{f(-q^2,-q^8)f(-q^4,-q^6)} = \frac{G(q)G(-q)H(q)H(-q)}{G(q^2)H(q^2)}. \tag{5.3.25}$$

By the Jacobi Triple Product Identity,

$$\begin{aligned}f(q^2,q^8)f(q^4,q^6) &= (-q^2;q^{10})_\infty(-q^4;q^{10})_\infty(-q^6;q^{10})_\infty(-q^8;q^{20})_\infty(q^{10};q^{10})^2_\infty\\ &= \frac{(-q^2;q^2)_\infty}{(-q^{10};q^{10})_\infty}(q^{10};q^{10})^2_\infty = \frac{(q^4;q^4)_\infty(q^{10};q^{10})^3_\infty}{(q^2;q^2)_\infty(q^{20};q^{20})_\infty}\\ &= \frac{f(-q^4)f^3(-q^{10})}{f(-q^2)f(-q^{20})}.\end{aligned} \tag{5.3.26}$$

By (5.3.21),

$$f(-q^2,-q^8)f(-q^4,-q^6) = f(-q^2,-q^4)f(-q^{10},-q^{20}) = f(-q^2)f(-q^{10}). \tag{5.3.27}$$

Combine (5.3.26) and (5.3.27) to see that

$$\frac{f(q^2,q^8)f(q^4,q^6)}{f(-q^2,-q^8)f(-q^4,-q^6)} = \frac{f(-q^4)f^2(-q^{10})}{f^2(-q^2)f(-q^{20})}. \tag{5.3.28}$$

Apply (2.3.6) followed by Lemma 1.0.1 to find that

$$\frac{G(q)G(-q)H(q)H(-q)}{G(q^2)H(q^2)} = \frac{f(-q^5)f(q^5)f(-q^2)}{f(-q)f(q)f(-q^{10})} = \frac{f(-q^4)f^2(-q^{10})}{f^2(-q^2)f(-q^{20})}. \tag{5.3.29}$$

Using (5.3.28) and (5.3.29), we readily deduce the truth of (5.3.25), and thus complete the proof of (5.3.2). □

Next, we record several natural and beautiful corollaries of our work.

Corollary 5.3.2. *We have*

$$\frac{G(q)H(-q) - G(-q)H(q)}{G(q)H(q) - G(-q)H(-q)} = \frac{\phi(-q^2)\phi(-q^{20})}{\phi(-q^4)\phi(-q^{10})} \tag{5.3.30}$$

and

$$\frac{G(q)H(-q)+G(-q)H(q)}{G(q)H(q)+G(-q)H(-q)}=\frac{\phi(-q^4)\phi(-q^{10})}{\chi(-q^{10})\chi(-q^{20})}. \tag{5.3.31}$$

Proof. To prove (5.3.30), rearrange (5.3.1) to see that

$$\frac{G(q)H(-q)-G(-q)H(q)}{G(q)H(q)-G(-q)H(-q)}=\frac{G(-q^2)H(-q^2)}{G(q^2)H(q^2)}=\frac{f(q^{10})f(-q^2)}{f(q^2)f(-q^{10})}, \tag{5.3.32}$$

where we have employed (2.3.6) to obtain the right-most expression in (5.3.32). Utilizing Lemma 1.0.1, we readily complete the proof. Similarly (5.3.31) follows from (5.3.2); we omit the details. □

Corollary 5.3.3. *We have*

$$G(q)H(q)-G(-q)H(-q)=2q\frac{\psi(q^{10})}{f(-q^2)}\frac{f(q^2)f(-q^{10})}{f(-q^2)f(q^{10})}, \tag{5.3.33}$$

$$G(q)H(q)+G(-q)H(-q)=2\frac{\psi(q^2)}{f(-q^2)}\frac{\chi(-q^{10})f(q^{10})}{\chi(-q^2)f(q^2)}. \tag{5.3.34}$$

Proof. To prove (5.3.34), rewrite (5.3.2) and apply (5.3.29) with q replaced by q^2, Theorem 2.5.1(iv), and (1.0.13) in order to deduce that

$$\begin{aligned}
&G(q)H(q)+G(-q)H(-q)\\
&\quad=\frac{G(q^2)H(q^2)G(-q^2)H(-q^2)}{G(q^4)H(q^4)}\left[G(q)H(-q)+G(-q)H(q)\right]\\
&\quad=\frac{f(-q^8)f^2(-q^{20})}{f^2(-q^4)f(-q^{40})}\left[2\frac{\psi(q^2)}{f(-q^2)}\right]\\
&\quad=2\frac{\psi(q^2)}{f(-q^2)}\frac{\chi(-q^{10})f(q^{10})}{\chi(-q^2)f(q^2)}.
\end{aligned}\tag{5.3.35}$$

Similarly, (5.3.33) follows from (5.3.1), (5.3.11), and Theorem 2.5.1(iii). □

We also have the following elegant theta function identities.

Corollary 5.3.4. *We have*

$$f(-q^5)f(q)-f(q^5)f(-q)=2q\phi(-q^4)\psi(-q^{10}), \tag{5.3.36}$$

$$f(-q^5)f(q)+f(q^5)f(-q)=2\psi(-q^2)\phi(-q^{20}). \tag{5.3.37}$$

Proof. To prove (5.3.36), apply (2.3.6) to (5.3.33). Thus,

$$\frac{f(-q^5)}{f(-q)}-\frac{f(q^5)}{f(q)}=2q\frac{\psi(q^{10})}{f(-q^2)}\frac{f(q^2)f(-q^{10})}{f(-q^2)f(q^{10})}, \tag{5.3.38}$$

whence we deduce that

$$f(-q^5)f(q)-f(q^5)f(-q)=2qf(q)f(-q)\frac{\psi(q^{10})}{f(-q^2)}\frac{f(q^2)f(-q^{10})}{f(-q^2)f(q^{10})}. \tag{5.3.39}$$

With the help of Lemma 1.0.1, we see that the right-hand sides of (5.3.36) and (5.3.39) are equal, and thus we deduce (5.3.36). Similarly, we deduce (5.3.37) from (5.3.34). This completes the proof. □

5.4 An Identity for the Level 9

The following theorem was conjectured by Koike [63] and first proved by Bringmann and Swisher [29] using the theory of modular forms.

Theorem 5.4.1.

$$\big(G(q^9)G(q)+q^2H(q^9)H(q)\big)^6=\frac{f^3(-q)}{f^3(-q^9)}+27q^2\frac{f^3(-q^9)}{f^3(-q)}+9q. \tag{5.4.1}$$

Proof. Recall (5.2.3), namely,

$$G(q^9)G(q)+q^2H(q^9)H(q)=\frac{f^2(-q^3)}{f(-q)f(-q^9)}. \tag{5.4.2}$$

Raising both sides of (5.4.2) to the sixth power and comparing the result with (5.4.1), we see that it suffices to show that

$$\frac{f^3(-q)}{f^3(-q^9)}+27q^2\frac{f^3(-q^9)}{f^3(-q)}+9q=\frac{f^{12}(-q^3)}{f^6(-q)f^6(-q^9)}. \tag{5.4.3}$$

To prove (5.4.3), we employ a second result of Ramanujan, namely [14, p. 345, Entry 1(iv)],

$$3+\frac{f^3(-q^{1/3})}{q^{1/3}f^3(-q^3)}=\left(27+\frac{f^{12}(-q)}{qf^{12}(-q^3)}\right)^{1/3}. \tag{5.4.4}$$

Replace $q^{1/3}$ by q in (5.4.4) and cube both sides. Multiply by $q^3f^6(-q^9)/f^6(-q)$ and simplify to deduce (5.4.3). This completes the proof. $\square$

5.5 An Identity for the Level 56

The next result was conjectured by Koike [63] and first proved in [29].

Theorem 5.5.1.

$$\begin{aligned}&\left\{G(q^{56})H(q)-q^{11}H(q^{56})G(q)\right\}\left\{G(q^7)G(q^8)+q^3H(q^7)H(q^8)\right\}\\&=\frac{\chi(-q^4)\chi(-q^{28})}{\chi(-q)\chi(-q^7)}-q.\end{aligned} \tag{5.5.1}$$

Proof. Multiplying (5.2.12) with (5.2.13) and replacing q^2 by q, we deduce that

$$\begin{aligned}&\left\{G(q^{56})H(q)-q^{11}H(q^{56})G(q)\right\}\left\{G(q^7)G(q^8)+q^3H(q^7)H(q^8)\right\}\\&=\frac{1}{4}\chi(-q^2)\chi(-q^{14})\left\{\chi(q^{1/2})\chi(-q^{7/2})+\chi(-q^{1/2})\chi(q^{7/2})\right\}^2.\end{aligned} \tag{5.5.2}$$

Comparing (5.5.2) with (5.5.1), we see that it is sufficient to show that

$$\begin{aligned}&\frac{1}{4}\chi(-q^2)\chi(-q^{14})\left\{\chi(q^{1/2})\chi(-q^{7/2})+\chi(-q^{1/2})\chi(q^{7/2})\right\}^2\\&=\frac{\chi(-q^4)\chi(-q^{28})}{\chi(-q)\chi(-q^7)}-q.\end{aligned} \tag{5.5.3}$$

By Lemma 1.0.1 (see also [14, p. 39, Entry 24(iii)] for the second equality), we readily see that

$$\chi(q)=\sqrt{\frac{\phi(q)}{f(-q^2)}}=\frac{f(-q^2)}{\psi(-q)}. \tag{5.5.4}$$

For the expression inside the brackets in (5.5.3), we apply the first equality in (5.5.4) with q replaced by $\pm q^{1/2}$ and by $\pm q^{7/2}$. We also apply the second representation for $\chi(q)$ in (5.5.4) to the two χ functions outside the brackets on the left-hand side of (5.5.3). We thus write (5.5.3) in the equivalent form

$$\begin{aligned}&\frac{1}{4}\frac{f(-q^4)f(-q^{28})}{\psi(q^2)\psi(q^{14})}\left\{\sqrt{\frac{\phi(q^{1/2})\phi(-q^{7/2})}{f(-q)f(-q^7)}}+\sqrt{\frac{\phi(-q^{1/2})\phi(q^{7/2})}{f(-q)f(-q^7)}}\right\}^2\\&=\frac{\chi(-q^4)\chi(-q^{28})}{\chi(-q)\chi(-q^7)}-q.\end{aligned} \tag{5.5.5}$$

By Lemma 1.0.1 (see also [14, p. 40, Entry25(iii)]),

$$\phi(q)\phi(-q)=\phi^2(-q^2). \tag{5.5.6}$$

Multiplying (5.5.5) by $\psi(q^2)\psi(q^{14})f(-q)f(-q^7)/\left(f(-q^4)f(-q^{28})\right)$, expanding out the square on the left-hand side of (5.5.5), and simplifying with the use of (5.5.6), we find that

$$\begin{aligned}&\frac{1}{4}\left\{\phi(q^{1/2})\phi(-q^{7/2})+\phi(-q^{1/2})\phi(q^{7/2})\right\}+\frac{1}{2}\phi(-q)\phi(-q^7)\\&\qquad=\frac{\psi(q^2)\psi(q^{14})f(-q)f(-q^7)}{f(-q^4)f(-q^{28})}\left\{\frac{\chi(-q^4)\chi(-q^{28})}{\chi(-q)\chi(-q^7)}-q\right\}.\end{aligned} \tag{5.5.7}$$

With the help of Lemma 1.0.1, we deduce that

$$\frac{\psi(q^2)\psi(q^{14})f(-q)f(-q^7)}{f(-q^4)f(-q^{28})}\cdot\frac{\chi(-q^4)\chi(-q^{28})}{\chi(-q)\chi(-q^7)}=\phi(-q^4)\phi(-q^{28}) \tag{5.5.8}$$

and

$$\frac{\psi(q^2)\psi(q^{14})f(-q)f(-q^7)}{f(-q^4)f(-q^{28})}=\psi(-q)\psi(-q^7). \tag{5.5.9}$$

Employing (5.5.8) and (5.5.9) in (5.5.7) and then replacing q^2 by q, we deduce that (5.5.3) is equivalent to the theta function identity

$$\begin{aligned}&\frac{1}{4}\left\{\phi(q^{1/4})\phi(-q^{7/4})+\phi(-q^{1/4})\phi(q^{7/4})\right\}+\frac{1}{2}\phi(-q^{1/2})\phi(-q^{7/2})\\&\qquad=\phi(-q^2)\phi(-q^{14})-q^{1/2}\psi(-q^{1/2})\psi(-q^{7/2}).\end{aligned} \tag{5.5.10}$$

We prove (5.5.10). To that end, we first convert (5.5.10) into a modular equation. Let β be of the seventh degree in α. Employing [14, pp. 122–123, Entries 10(iii), (vii)–(ix); Entry 11(vii)], we find after performing some minor simplifications that (5.5.10) is equivalent to the modular equation

$$\begin{aligned}&\frac{1}{4}\left\{\left(1+\alpha^{1/4}\right)\left(1-\beta^{1/4}\right)+\left(1-\alpha^{1/4}\right)\left(1+\beta^{1/4}\right)\right\}\\&+\frac{1}{2}\left(1-\sqrt{\alpha}\right)^{1/2}\left(1-\sqrt{\beta}\right)^{1/2}\end{aligned}$$

$$= (1-\alpha)^{1/8} (1-\beta)^{1/8} - \frac{1}{\sqrt{2}} (1-\sqrt{\alpha})^{1/4} \left(1-\sqrt{\beta}\right)^{1/4} (\alpha\beta)^{1/16}. \quad (5.5.11)$$

To prove (5.5.11), we require two modular equations of degree 7 stated by Ramanujan, namely [14, p. 314, Entry 19(i)]

$$(\alpha\beta)^{1/8} + \{(1-\alpha)(1-\beta)\}^{1/8} = 1, \quad (5.5.12)$$

$$\left(\frac{1}{2}\left(1+(\alpha\beta)^{1/2}+\{(1-\alpha)(1-\beta)\}^{1/2}\right)\right)^{1/2} = 1 - \{\alpha\beta(1-\alpha)(1-\beta)\}^{1/8}. \quad (5.5.13)$$

Ramanujan recorded a large number of elegant modular equations of degree seven in his notebooks [14, pp. 314–315], [7, pp. 387–388, 391–392]. The seventh-order modular equation (5.5.12) is due originally to C. Guetzlaff [47] in 1834. H. Schröter in 1854 [86], [87], [88] and E. Fiedler [40] in 1885 also proved this modular equation. More complicated modular equations of degree seven have been discovered by L. Schläfli [84], F. Klein [61], L. Sohncke [90], [92], and R. Russell [83].

Beginning with (5.5.12), multiply through by $\{(1-\alpha)(1-\beta)\}^{1/8}$, apply (5.5.12) to the right-hand side of the resulting equation, and then apply (5.5.13) and rearrange the results. We thus deduce the sequence of equalities

$$\{(1-\alpha)(1-\beta)\}^{1/4} + \{\alpha\beta(1-\alpha)(1-\beta)\}^{1/8} = \{(1-\alpha)(1-\beta)\}^{1/8},$$

$$\{(1-\alpha)(1-\beta)\}^{1/4} + \{\alpha\beta(1-\alpha)(1-\beta)\}^{1/8} = 1-(\alpha\beta)^{1/8},$$

and

$$\{(1-\alpha)(1-\beta)\}^{1/4} + (\alpha\beta)^{1/8} = \sqrt{\frac{1}{2}\left(1+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}\right)}. \quad (5.5.14)$$

Now,

$$\frac{1}{2}\left[(1+\sqrt{\alpha})^{1/2}\left(1+\sqrt{\beta}\right)^{1/2} + (1-\sqrt{\alpha})^{1/2}\left(1-\sqrt{\beta}\right)^{1/2}\right] = \sqrt{\frac{1}{2}\left(1+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}\right)}, \quad (5.5.15)$$

as can readily be seen by squaring both sides. Inserting the left-side of (5.5.15) into (5.5.14), we deduce that

$$\{(1-\alpha)(1-\beta)\}^{1/4} + (\alpha\beta)^{1/8} = \frac{1}{2}\left[(1+\sqrt{\alpha})^{1/2}\left(1+\sqrt{\beta}\right)^{1/2} + (1-\sqrt{\alpha})^{1/2}\left(1-\sqrt{\beta}\right)^{1/2}\right]. \quad (5.5.16)$$

Observe that by squaring and rearranging (5.5.17), we readily deduce that (5.5.16) is equivalent to the beautiful identity

$$\sqrt{2}\,(\alpha\beta)^{1/16} = \left[(1+\sqrt{\alpha})\left(1+\sqrt{\beta}\right)\right]^{1/4} - \left[(1-\sqrt{\alpha})\left(1-\sqrt{\beta}\right)\right]^{1/4}. \quad (5.5.17)$$

Multiply (5.5.17) by $-\frac{1}{2}\cdot(1-\sqrt{\alpha})^{1/4}(1-\sqrt{\beta})^{1/4}$, add $(1-\alpha)^{1/4}(1-\beta)^{1/4}$ to both sides of the resulting equation, and simplify. We thus deduce that

$$\begin{aligned}&\{(1-\alpha)(1-\beta)\}^{1/8}\cdot\{(1-\alpha)(1-\beta)\}^{1/8}\\&-\frac{1}{\sqrt{2}}\left(1-\sqrt{\alpha}\right)^{1/4}\left(1-\sqrt{\beta}\right)^{1/4}(\alpha\beta)^{1/16}\\&\qquad=\frac{1}{2}(1-\alpha)^{1/4}(1-\beta)^{1/4}+\frac{1}{2}\left(1-\sqrt{\alpha}\right)^{1/2}\left(1-\sqrt{\beta}\right)^{1/2}.\end{aligned}\tag{5.5.18}$$

By (5.5.12), the first term on the left-hand side of (5.5.18) is equal to

$$\{(1-\alpha)(1-\beta)\}^{1/8}\cdot\left[1-(\alpha\beta)^{1/8}\right].\tag{5.5.19}$$

Inserting (5.5.19) into (5.5.18) and rearranging, we find that

$$\begin{aligned}&\frac{1}{2}\{(1-\alpha)(1-\beta)\}^{1/4}+\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/8}\\&+\frac{1}{2}\left(1-\sqrt{\alpha}\right)^{1/2}\left(1-\sqrt{\beta}\right)^{1/2}\\&=\{(1-\alpha)(1-\beta)\}^{1/8}-\frac{1}{\sqrt{2}}\left[\left(1-\sqrt{\alpha}\right)\left(1-\sqrt{\beta}\right)\right]^{1/4}(\alpha\beta)^{1/16}.\end{aligned}\tag{5.5.20}$$

Squaring (5.5.12), dividing all terms by 1/2 and rearranging, we deduce that

$$\frac{1}{2}\{(1-\alpha)(1-\beta)\}^{1/4}+\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/8}=\frac{1}{2}-\frac{1}{2}(\alpha\beta)^{1/4}.\tag{5.5.21}$$

Employing (5.5.21), we find that the left-hand side of (5.5.20) is equal to

$$\frac{1}{2}-\frac{1}{2}(\alpha\beta)^{1/4}+\frac{1}{2}\left(1-\sqrt{\alpha}\right)^{1/2}\left(1-\sqrt{\beta}\right)^{1/2},$$

which, after a modicum of algebra, is readily seen to be equal to

$$\begin{aligned}&\frac{1}{4}\left[\left(1+\alpha^{1/4}\right)\left(1-\beta^{1/4}\right)+\left(1-\alpha^{1/4}\right)\left(1+\beta^{1/4}\right)\right]\\&+\frac{1}{2}\left(1-\sqrt{\alpha}\right)^{1/2}\left(1-\sqrt{\beta}\right)^{1/2}.\end{aligned}\tag{5.5.22}$$

Replacing the left-hand side of (5.5.20) with (5.5.22), we deduce (5.5.11). This completes the proof. □

5.6 Identities for the Level 84

The next theorem was discovered empirically on a computer by Koike [63] and independently by Somos [94].

Theorem 5.6.1.

$$\frac{G(q^{84})G(q)+q^{17}H(q^{84})H(q)}{G(q^{21})G(q^4)+q^5H(q^{21})H(q^4)}=\frac{G(q^{28})H(q^3)-q^5H(q^{28})G(q^3)}{G(q^{12})H(q^7)-qH(q^{12})G(q^7)}\tag{5.6.1}$$

and

$$G(q^{84})G(q)+q^{17}H(q^{84})H(q)=\frac{G(q^{42})H(q^2)-q^8H(q^{42})G(q^2)}{G(q^{12})H(q^7)-qH(q^{12})G(q^7)}.\tag{5.6.2}$$

Proof. In the notation of Theorem 5.2.3, (5.6.1) reads as

$$\frac{C(q)}{W(q)} = \frac{Z(q)}{B(-q)}. \tag{5.6.3}$$

Applying the first equality in (5.2.18), the third equality of (5.2.19) with q^2 replaced by $-q^2$, and the first equality of (5.2.19) with q replaced by q^2, we readily deduce that

$$C(q^2)B(-q^2) = Y(-q^2)V(-q^2) = W(-q^2)Z(-q^2) = W(q^2)Z(q^2). \tag{5.6.4}$$

Replacing q^2 by q in (5.6.4) and rearranging, we arrive at (5.6.3). This completes the proof of (5.6.1).

To prove (5.6.2), we first define

$$T(q) := G(q^{21})H(q) - q^4 H(q^{21})G(q). \tag{5.6.5}$$

Then, using the notation of Theorem 5.2.3 and (5.6.5), we cast (5.6.2) in the equivalent form

$$B(-q^2)C(q^2) = T(q^4). \tag{5.6.6}$$

By repeated applications of Theorem 5.2.3, we conclude that

$$\begin{aligned} B(-q^2) &= \frac{f(-q^2)f(-q^{84})}{f(-q^{12})f(-q^{14})}V(-q^2) \\ &= \frac{f(-q^2)f(-q^{84})}{f(-q^{12})f(-q^{14})} \cdot \frac{f(q^6)f(-q^{28})}{f(-q^4)f(q^{42})}Z(-q^2) \\ &= \frac{f(-q^2)f(-q^{84})}{f(-q^{12})f(-q^{14})} \cdot \frac{f(q^6)f(-q^{28})}{f(-q^4)f(q^{42})} \cdot \frac{f(-q^8)f(-q^{84})}{f(-q^{12})f(-q^{56})}W(q^2). \end{aligned} \tag{5.6.7}$$

Recall from Lemma 1.0.1 that

$$f(q) = \frac{f^3(-q^2)}{f(-q)f(-q^4)}. \tag{5.6.8}$$

Hence, employing (5.6.8) in (5.6.7), we find that

$$\begin{aligned} B(-q^2)C(q^2) &= W(q^2)C(q^2) \\ &\quad \times \left[\frac{f(-q^2)f(-q^8)f(-q^{12})f(-q^{28})f(-q^{42})f(-q^{168})}{f(-q^4)f(-q^6)f(-q^{14})f(-q^{24})f(-q^{56})f(-q^{84})}\right]. \end{aligned} \tag{5.6.9}$$

By Theorem 5.2.4 with q replaced by q^2,

$$\begin{aligned} W(q^2)C(q^2) = \frac{1}{2q^2}\Bigg\{ &\frac{f^3(-q^4)f^2(-q^6)f(-q^{24})f(-q^{28})}{f^2(-q^2)f^2(-q^8)f^2(-q^{12})f(-q^{168})} \\ &- \frac{f(-q^{12})f^2(-q^{14})f(-q^{56})f^3(-q^{84})}{f(-q^8)f^2(-q^{28})f^2(-q^{42})f^2(-q^{168})}\Bigg\}. \end{aligned} \tag{5.6.10}$$

Combining (5.6.9) with (5.6.10), simplifying with the help of Lemma 1.0.1, and finally applying (5.2.9) and the definition of $T(q)$, we deduce that

$$B(-q^2)C(q^2) = \frac{1}{2q^2}\Bigg\{ \frac{f^2(-q^4)}{f(-q^2)f(-q^8)} \cdot \frac{f(-q^6)}{f(-q^{12})} \cdot \frac{f^2(-q^{28})}{f(-q^{14})f(-q^{56})} \cdot \frac{f(-q^{42})}{f(-q^{84})}$$

$$
\begin{aligned}
&\qquad - \frac{f(-q^2)}{f(-q^4)} \cdot \frac{f^2(-q^{12})}{f(-q^6)f(-q^{24})} \cdot \frac{f(-q^{14})}{f(-q^{28})} \cdot \frac{f^2(-q^{84})}{f(-q^{42})f(-q^{168})} \Bigg\} \\
&= \frac{1}{2q^2}\{\chi(q^2)\chi(-q^6)\chi(q^{14})\chi(-q^{42}) \\
&\qquad - \chi(-q^2)\chi(q^6)\chi(-q^{14})\chi(q^{42})\} \\
&= T(q^4). \qquad (5.6.11)
\end{aligned}
$$

This completes the proof. □

We record some interesting factorization theorems as corollaries of Theorem 5.6.1 and the work in [23].

Corollary 5.6.2. *The following factorizations hold:*

$$
\begin{aligned}
&\left\{G(q^3)G(-q^7) - q^2H(q^3)H(-q^7)\right\}\left\{G(q^{21})H(-q) + q^4H(q^{21})G(-q)\right\} \\
&= \left\{G(-q^3)G(q^7) - q^2H(-q^3)H(q^7)\right\}\left\{G(-q^{21})H(q) + q^4H(-q^{21})G(q)\right\} \\
&= G(q^{42})H(q^2) - q^8H(q^{42})G(q^2) \\
&= G(q^6)G(q^{14}) + q^4H(q^6)H(q^{14}). \qquad (5.6.12)
\end{aligned}
$$

Proof. Recall the notation of Theorem 5.2.3 as well as the definition of $T(q)$, (5.6.5). By Theorem 5.2.3,

$$
\begin{aligned}
B(-q^2)C(q^2) &= V(-q^2)Y(-q^2) = W(-q^2)Z(-q^2) \\
&= W(q^2)Z(q^2) = V(q^2)Y(q^2). \qquad (5.6.13)
\end{aligned}
$$

Combining (5.6.13) with (5.6.6), we find that

$$
V(q^2)Y(q^2) = V(-q^2)Y(-q^2) = T(q^4). \qquad (5.6.14)
$$

Replacing q^2 by q in (5.6.14), we deduce the first two equalities in (5.6.12). The third equality in (5.6.12) is simply (5.2.8) with q replaced by q^2. □

Corollary 5.6.3. *The following factorizations hold:*

$$
\begin{aligned}
&\frac{1}{2q^2}\frac{\chi(-q^{12})\chi(-q^{84})}{\chi(q^2)\chi(q^{14})}\left\{\chi(q)\chi(-q^{21}) - \chi(-q)\chi(q^{21})\right\} \\
&\qquad\qquad \times \left\{\chi(q^3)\chi(-q^7) - \chi(-q^3)\chi(q^7)\right\} \\
&= \frac{q^2\chi(-q^4)\chi(-q^{28})}{2\chi(q^6)\chi(q^{42})}\left\{\chi(q)\chi(-q^{21}) + \chi(-q)\chi(q^{21})\right\} \\
&\qquad\qquad \times \left\{\chi(q^3)\chi(-q^7) + \chi(-q^3)\chi(q^7)\right\} \\
&= \{\chi(q^2)\chi(-q^6)\chi(q^{14})\chi(-q^{42}) - \chi(-q^2)\chi(q^6)\chi(-q^{14})\chi(q^{42})\}. \qquad (5.6.15)
\end{aligned}
$$

Proof. Employing (5.2.14), (5.2.15), (5.6.14), and (5.2.9), we readily deduce that the first and third expressions in Corollary 5.6.3 are equal. Similarly, from (5.2.16), (5.2.17), (5.6.13), (5.6.14), and (5.2.9), we verify that the second and third expressions in Corollary 5.6.3 are equal. This completes the proof. □

5.7 An Identity for the Level 96

Koike [63] conjectured Theorem 5.7.1, and Bringmann and Swisher [29] gave the first proof.

Theorem 5.7.1. *We have*

$$\left\{G(q^{48})G(q^2)+q^{10}H(q^{48})H(q^2)\right\}\left\{G(q^{32})G(q^3)+q^7H(q^{32})H(q^3)\right\}$$
$$=\left\{G(q^{96})H(q)-q^{19}H(q^{96})G(q)\right\}.$$

Proof. Let us define $K(q)$, $L(q)$, and $M(q)$ by

$$\begin{aligned}K(q)&:=G(q^{24})G(q)+q^5H(q^{24})H(q),\\ L(q)&:=G(q^{96})H(q)-q^{19}H(q^{96})G(q),\\ M(q)&:=G(q^{32})G(q^3)+q^7H(q^{32})H(q^3).\end{aligned}$$

Then a slight rearrangement of Theorem 5.7.1 reads

$$K(q^2)=\frac{L(q)}{M(q)}. \tag{5.7.1}$$

Employing (5.2.1) and (5.2.2), we find that

$$\begin{aligned}L(q)&=\frac{f(-q^8)}{f(-q^2)}\{G(q^{96})\left(q^3H(q^{16})+G(-q^4)\right)\\ &\qquad\qquad -q^{19}H(q^{96})\left(G(q^{16})+qH(-q^4)\right)\}\\ &=\frac{f(-q^8)}{f(-q^2)}\{q^3\left(G(q^{96})H(q^{16})-q^{16}H(q^{96})G(q^{16})\right)\\ &\qquad\qquad +\left(G(q^{96})G(-q^4)-q^{20}H(q^{96})H(-q^4)\right)\}.\end{aligned} \tag{5.7.2}$$

Applying (5.2.5) with q replaced by q^{16} and (5.2.7) with q replaced by $-q^4$, we conclude from (5.7.2) that

$$L(q)=\frac{f(-q^8)}{f(-q^2)}\left\{q^3\frac{\chi(-q^{16})}{\chi(-q^{48})}+\frac{\chi(q^{12})\chi(-q^{48})}{\chi(q^4)\chi(-q^{16})}\right\}. \tag{5.7.3}$$

In the definition of $M(q)$ above, employ (5.2.1) and (5.2.2), each with q replaced by q^3. Then apply (5.2.4) with q replaced by q^{16} and (5.2.6) with q replaced by $-q^4$, arguing as in (5.7.3), in order to conclude that

$$M(q)=\frac{f(-q^{24})}{f(-q^6)}\left\{q^3\frac{\chi(q^4)\chi(-q^{16})}{\chi(q^{12})\chi(-q^{48})}+\frac{\chi(-q^{48})}{\chi(-q^{16})}\right\}. \tag{5.7.4}$$

A slight simplification of (5.7.4) yields

$$M(q)=\frac{f(-q^{24})\chi(q^4)}{f(-q^6)\chi(q^{12})}\left\{q^3\frac{\chi(-q^{16})}{\chi(-q^{48})}+\frac{\chi(q^{12})\chi(-q^{48})}{\chi(q^4)\chi(-q^{16})}\right\}. \tag{5.7.5}$$

Combining (5.7.3) and (5.7.5), we arrive at

$$\frac{L(q)}{M(q)}=\frac{f(-q^6)f(-q^8)\chi(q^{12})}{f(-q^2)f(-q^{24})\chi(q^4)}. \tag{5.7.6}$$

By (5.2.7) with q replaced by q^2, we see that

$$K(q^2)=\frac{\chi(-q^6)\chi(-q^{24})}{\chi(-q^2)\chi(-q^8)}. \tag{5.7.7}$$

With the use of Lemma 1.0.1, we readily verify that

$$\frac{f(-q^6)f(-q^8)\chi(q^{12})}{f(-q^2)f(-q^{24})\chi(q^4)} = \frac{\chi(-q^6)\chi(-q^{24})}{\chi(-q^2)\chi(-q^8)}. \tag{5.7.8}$$

Combining (5.7.6), (5.7.7), and (5.7.8), we deduce (5.7.1). This completes the proof. □

5.8 Identities for the Level 104

Next, we prove relations for the level 104. Our primary result is Theorem 5.8.1, which was originally conjectured by Koike [63] and proved first in [29]. The proof of this identity is harder than the proofs considered thus far. At the end, we record some interesting corollaries yielded by our methods.

Theorem 5.8.1. *We have*

$$\begin{aligned}&\left\{G(q^{13})H(q^8) - qH(q^{13})G(q^8)\right\}\left\{G(q)G(q^{104}) + q^{21}H(q)H(q^{104})\right\}\\ &= G(q^{52})H(q^2) - q^{10}H(q^{52})H(q^2).\end{aligned} \tag{5.8.1}$$

Before we begin, we establish the following lemmas.

Lemma 5.8.2. *We have*

$$f(-q,-q^9) = G(-q)H(q^4)f(-q^2), \tag{5.8.2}$$

$$f(-q^3,-q^7) = H(-q)G(q^4)f(-q^2), \tag{5.8.3}$$

$$\phi(-q^5) = \left[G(-q)G(q^4) + qH(-q)H(q^4)\right] f(-q^2), \tag{5.8.4}$$

$$\frac{G(q)}{H(q)}f(-q^5) = f(-q^7,-q^8) + qf(-q^2,-q^{13}), \tag{5.8.5}$$

$$\frac{H(q)}{G(q)}f(-q^5) = f(-q^4,-q^{11}) - qf(-q,-q^{14}), \tag{5.8.6}$$

Proof. Relations (5.8.2)–(5.8.4) follow from replacing q by $-q$ in (2.5.25), (2.5.26), and Theorem 2.5.1(i), respectively.

By [14, p. 379, Entry 10(i),(ii)], we know that

$$f(-q^7,-q^8) + qf(-q^2,-q^{13}) = \frac{f(-q^2,-q^3)}{f(-q,-q^4)}f(-q^5) \tag{5.8.7}$$

and

$$f(-q^4,-q^{11}) - qf(-q,-q^{14}) = \frac{f(-q,-q^4)}{f(-q^2,-q^3)}f(-q^5). \tag{5.8.8}$$

Applying both (2.1.2) and (2.1.3) to each of (5.8.7) and (5.8.8), we deduce (5.8.5) and (5.8.6), respectively. This completes the proof. □

Lemma 5.8.3. *The following relations hold:*

$$
\begin{aligned}
&\{G(q^{26})H(-q)+q^5H(q^{26})G(-q)\}\\
&\quad\times\{G(q^4)G(q^{26})+q^6H(q^4)H(q^{26})\}f(-q^2)f(-q^{26})\\
&=f(-q^3,-q^7)\{f(-q^{182},-q^{208})+q^{26}f(-q^{52},-q^{338})\}\\
&\quad+q^{11}f(-q,-q^9)\{f(-q^{104},-q^{286})-q^{26}f(-q^{26},-q^{364})\}+q^5\phi(-q^5)f(-q^{130})
\end{aligned}
\tag{5.8.9}
$$

and

$$
\begin{aligned}
&-\{G(q^{52})H(q^2)-q^{10}G(q^2)H(q^{52})\}\\
&\quad\times\{G(q^2)G(-q^{13})-q^3H(q^2)H(-q^{13})\}f(-q^2)f(-q^{26})\\
&=q^3f(-q^{39},-q^{91})\{f(-q^8,-q^{22})-q^2f(-q^2,-q^{28})\}\\
&\quad+q^{10}f(-q^{13},-q^{117})\{f(-q^{14},-q^{16})+q^2f(-q^4,-q^{26})\}-\phi(-q^{65})f(-q^{10}).
\end{aligned}
\tag{5.8.10}
$$

Proof. Applying (5.8.2)–(5.8.6) to the right-hand side of (5.8.9) and simplifying, we conclude that

$$
\begin{aligned}
&f(-q^3,-q^7)\{f(-q^{182},-q^{208})+q^{26}f(-q^{52},-q^{338})\}\\
&+q^{11}f(-q,-q^9)\{f(-q^{104},-q^{286})-q^{26}f(-q^{26},-q^{364})\}+q^5\phi(-q^5)f(-q^{130})\\
&=G(q^4)H(-q)f(-q^2)\left\{\frac{G(q^{26})}{H(q^{26})}f(-q^{130})\right\}\\
&\quad+q^{11}G(-q)H(q^4)f(-q^2)\left\{\frac{H(q^{26})}{G(q^{26})}f(-q^{130})\right\}\\
&\quad+q^5\left[G(-q)G(q^4)+qH(-q)H(q^4)\right]f(-q^2)f(-q^{130})\\
&=f(-q^2)f(-q^{130})\left\{\frac{H(-q)G(q^4)G(q^{26})}{H(q^{26})}+q^{11}\frac{G(-q)H(q^4)H(q^{26})}{G(q^{26})}\right\}\\
&\quad+q^5f(-q^2)f(-q^{130})\frac{G(q^{26})H(q^{26})}{G(q^{26})H(q^{26})}\left[G(-q)G(q^4)+qH(-q)H(q^4)\right]\\
&=\frac{f(-q^2)f(-q^{130})}{G(q^{26})H(q^{26})}\Big[\{G^2(q^{26})G(q^4)H(-q)+q^{11}H^2(q^{26})H(q^4)G(-q)\}\\
&\quad+q^5[G(-q)G(q^4)G(q^{26})H(q^{26})+qH(-q)H(q^4)G(q^{26})H(q^{26})]\Big]\\
&=\frac{f(-q^2)f(-q^{130})}{G(q^{26})H(q^{26})}\\
&\quad\times\left[\big(G(q^{26})H(-q)+q^5H(q^{26})G(-q)\big)\big(G(q^4)G(q^{26})+q^6H(q^4)H(q^{26})\big)\right]\\
&=f(-q^2)f(-q^{26})\\
&\quad\times\left[\big(G(q^{26})H(-q)+q^5H(q^{26})G(-q)\big)\big(G(q^4)G(q^{26})+q^6H(q^4)H(q^{26})\big)\right],
\end{aligned}
\tag{5.8.11}
$$

where we used (2.3.6) with q replaced by q^{26} in the final line. This establishes (5.8.9). The proof of (5.8.10) is analogous; we omit the details. □

Let us define, by (2.1.2) and (2.1.3),

$$g(q) := f(-q)G(q) = f(-q^2, -q^3) \quad \text{and} \quad h(q) := f(-q)H(q) = f(-q, -q^4). \tag{5.8.12}$$

Lemma 5.8.4. *The following relations hold:*

$$\frac{g(q)}{\chi(-q)} = f(-q^{13}, -q^{17}) + qf(-q^7, -q^{23}) \tag{5.8.13}$$

$$\frac{h(q)}{\chi(-q)} = f(-q^{11}, -q^{19}) + q^3 f(-q, -q^{29}) \tag{5.8.14}$$

Proof. In (1.0.19), replace q by q^5 and set $B = -q^4$ After one application of (1.0.5), we arrive at

$$\frac{f(-q^2, -q^8)}{f(-q, -q^9)} f(-q^{10}) = f(-q^{13}, -q^{17}) + qf(-q^7, -q^{23}). \tag{5.8.15}$$

With the help of (1.0.6), (1.0.10), and (5.8.12), the left-hand side of (5.8.15) is readily seen to be equal to $g(q)/\chi(-q)$. Hence, (5.8.13) is proved. The proof of (5.8.14) is analogous; we omit the details. □

Now, we prove Theorem 5.8.1.

Proof of Theorem 5.8.1. First, we derive identities for the two bracketed expressions in (5.8.1). Employing (5.2.1) and (5.2.2) in (5.2.11), we deduce that

$$\begin{aligned}
&\sqrt{\frac{\chi(-q^{13})}{\chi(-q)} - q\frac{\chi(-q)}{\chi(-q^{13})}} = G(q^{26})H(q) - q^5 H(q^{26})G(q) \\
&= \frac{f(-q^8)}{f(-q^2)} \left\{G(q^{26})\left(q^3 H(q^{16}) + G(-q^4)\right) - q^5 H(q^{26})\left(G(q^{16}) + qH(-q^4)\right)\right\} \\
&= \frac{f(-q^8)}{f(-q^2)} \{q^3 \left(G(q^{26})H(q^{16}) - q^2 H(q^{26})G(q^{16})\right) \\
&\qquad\qquad + \left(G(-q^4)G(q^{26}) - q^6 H(-q^4)H(q^{26})\right)\}.
\end{aligned} \tag{5.8.16}$$

Observe that the final expression in (5.8.16) gives us an odd-even dissection of the first expression in (5.8.16). Therefore, considering odd parts in (5.8.16), we find that

$$\begin{aligned}
&G(q^{26})H(q^{16}) - q^2 H(q^{26})G(q^{16}) \\
&= \frac{f(-q^2)}{2q^3 f(-q^8)} \left(\sqrt{\frac{\chi(-q^{13})}{\chi(-q)} - q\frac{\chi(-q)}{\chi(-q^{13})}} - \sqrt{\frac{\chi(q^{13})}{\chi(q)} + q\frac{\chi(q)}{\chi(q^{13})}}\right).
\end{aligned} \tag{5.8.17}$$

Similarly, starting from (5.2.11) and employing (5.2.1) and (5.2.2), both with q replaced by q^{13}, we see that

$$\begin{aligned}
&\sqrt{\frac{\chi(-q^{13})}{\chi(-q)} - q\frac{\chi(-q)}{\chi(-q^{13})}} = G(q^2)G(q^{13}) + q^3 H(q^2)H(q^{13}) \\
&= \frac{f(-q^{104})}{f(-q^{26})} \{G(q^2)\left(G(q^{208}) + q^{13} H(-q^{52})\right) \\
&\qquad\qquad + q^3 H(q^2)\left(q^{39} H(q^{208}) + G(-q^{52})\right)\}
\end{aligned}$$

$$= \frac{f(-q^{104})}{f(-q^{26})}\{\left(G(q^2)G(q^{208}) + q^{42}H(q^2)H(q^{208})\right)$$
$$+ q^3\left(G(-q^{52})H(q^2) + q^{10}H(-q^{52})G(q^2)\right)\}. \qquad (5.8.18)$$

We observe that the last expression in (5.8.18) yields another odd-even dissection of $\sqrt{\dfrac{\chi(-q^{13})}{\chi(-q)} - q\dfrac{\chi(-q)}{\chi(-q^{13})}}$. Considering even parts in (5.8.18), we deduce that

$$G(q^2)G(q^{208}) + q^{42}H(q^2)H(q^{208})$$
$$= \frac{f(-q^{26})}{2f(-q^{104})}\left(\sqrt{\frac{\chi(-q^{13})}{\chi(-q)} - q\frac{\chi(-q)}{\chi(-q^{13})}} + \sqrt{\frac{\chi(q^{13})}{\chi(q)} + q\frac{\chi(q)}{\chi(q^{13})}}\right). \qquad (5.8.19)$$

Multiplying together (5.8.17) and (5.8.19), we obtain

$$\{G(q^{26})H(q^{16}) - q^2H(q^{26})G(q^{16})\}\{G(q^2)G(q^{208}) + q^{42}H(q^2)H(q^{208})\}$$
$$= \frac{f(-q^2)f(-q^{26})}{4q^3f(-q^8)f(-q^{104})}\left[\left(\frac{\chi(-q^{13})}{\chi(-q)} - q\frac{\chi(-q)}{\chi(-q^{13})}\right) - \left(\frac{\chi(q^{13})}{\chi(q)} + q\frac{\chi(q)}{\chi(q^{13})}\right)\right]. \qquad (5.8.20)$$

Replacing q by q^2 in (5.8.1) and comparing the resulting equation with (5.8.20), we see that, in order to prove (5.8.1), it suffices to prove that

$$G(q^{104})H(q^4) - q^{20}H(q^{104})G(q^4)$$
$$= \frac{f(-q^2)f(-q^{26})}{4q^3f(-q^8)f(-q^{104})}\left[\left(\frac{\chi(-q^{13})}{\chi(-q)} - q\frac{\chi(-q)}{\chi(-q^{13})}\right) - \left(\frac{\chi(q^{13})}{\chi(q)} + q\frac{\chi(q)}{\chi(q^{13})}\right)\right]. \qquad (5.8.21)$$

By (5.8.12), $f(-q^4)f(-q^{104})$ times the left-hand side of (5.8.21) is equal to

$$g(q^{104})h(q^4) - q^{20}h(q^{104})g(q^4). \qquad (5.8.22)$$

Utilizing Lemma 1.0.1, we deduce that $f(-q^4)f(-q^{104})$ times the right-hand side of (5.8.21) is equal to

$$f(-q^4)f(-q^{104}) \cdot \frac{f(-q^2)f(-q^{26})}{4q^3f(-q^8)f(-q^{104})}$$
$$\times\left[\left(\frac{f(-q^{13})}{f(-q^{26})}\cdot\frac{f(-q^2)}{f(-q)} - q\frac{f(-q)}{f(-q^2)}\cdot\frac{f(-q^{26})}{f(-q^{13})}\right)\right.$$
$$\left.-\left(\frac{f^2(-q^{26})}{f(-q^{13})f(-q^{52})}\cdot\frac{f(-q)f(-q^4)}{f^2(-q^2)} + q\frac{f^2(-q^2)}{f(-q)f(-q^4)}\cdot\frac{f(-q^{13})f(-q^{52})}{f^2(-q^{26})}\right)\right]$$
$$= \frac{1}{4q^3}\frac{f(-q^4)}{f(-q^8)}\left[\frac{f^2(-q^2)}{f(-q)}\cdot f(-q^{13}) - qf(-q)\cdot\frac{f^2(-q^{26})}{f(-q^{13})}\right.$$
$$\left.-\frac{f(-q)f(-q^4)}{f(-q^2)}\cdot\frac{f^3(-q^{26})}{f(-q^{13})f(-q^{52})} - q\frac{f^3(-q^2)}{f(-q)f(-q^4)}\cdot\frac{f(-q^{13})f(-q^{52})}{f(-q^{26})}\right]$$
$$= \frac{1}{4q^3}\chi(-q^4)[\psi(q)f(-q^{13}) - qf(-q)\psi(q^{13}) - \psi(-q)f(q^{13}) - qf(q)\psi(-q^{13})]. \qquad (5.8.23)$$

Therefore, combining (5.8.22) and (5.8.23), we see that, in order to prove (5.8.1), it is sufficient to prove the beautiful identity

$$\begin{aligned}&\frac{1}{\chi(-q^4)}\left(g(q^{104})h(q^4)-q^{20}h(q^{104})g(q^4)\right)\\&=\frac{1}{4q^3}[\psi(q)f(-q^{13})-\psi(-q)f(q^{13})-qf(-q)\psi(q^{13})-qf(q)\psi(-q^{13})].\end{aligned}\tag{5.8.24}$$

To prove (5.8.24), we first provide alternative representations for the expressions $\psi(q)f(-q^{13})-\psi(-q)f(q^{13})$ and $-qf(-q)\psi(q^{13})-qf(q)\psi(-q^{13})$. To that end, we apply Theorem 1.0.3 with the parameters $\epsilon_1=0$, $\epsilon_2=1$, $a=1$, $b=q$, $c=q^{13}$, $d=q^{26}$, $\alpha=1$, $\beta=39$ and $m=40$. Accordingly, we deduce that

$$\begin{aligned}&f(1,q)f(-q^{13},-q^{26})\\&=\sum_{r=-20}^{19}(-1)^r q^{13r(3r-1)/2}f(-q^{26-39r},-q^{14+39r})f(-q^{806-39r},-q^{754+39r}).\end{aligned}\tag{5.8.25}$$

Applying Theorem 1.0.3 with the same set of parameters as for (5.8.25), except now with $\epsilon_1=1$ and $\epsilon_2=0$, we deduce that

$$\begin{aligned}&f(-1,-q)f(q^{13},q^{26})\\&=\sum_{r=-20}^{19}q^{13r(3r-1)/2}f(-q^{26-39r},-q^{14+39r})f(-q^{806-39r},-q^{754+39r}).\end{aligned}\tag{5.8.26}$$

Combining (5.8.25)–(5.8.26), along with the identities obtained from them by replacing q by $-q$, we see that

$$\begin{aligned}&f(1,q)f(-q^{13},-q^{26})-f(1,-q)f(q^{13},-q^{26})\\&+f(-1,-q)f(q^{13},q^{26})-f(-1,q)f(-q^{13},q^{26})=\sum_{i=0}^{3}S_i,\end{aligned}\tag{5.8.27}$$

where

$$\begin{aligned}S_i=\sum_{\substack{-20\le r\le 19\\ r\equiv i \pmod 4}}\Bigg[&\{((-1)^r+1)\,q^{13r(3r-1)/2}\\&\times f(-q^{26-39r},-q^{14+39r})f(-q^{806-39r},-q^{754+39r})\}\\&-\{((-1)^r+1)\,(-q)^{13r(3r-1)/2}\\&\times f(-(-q)^{26-39r},-(-q)^{14+39r})f(-(-q)^{806-39r},-(-q)^{754+39r})\}\Bigg].\end{aligned}\tag{5.8.28}$$

It is easy to show directly that

$$S_0=S_1=S_3=0.\tag{5.8.29}$$

Indeed, for $i=0$, replacing r by $4s$, we see that

$$S_0=\sum_{\substack{-20\le r\le 19\\ r\equiv 0 \pmod 4}}\Bigg[\{((-1)^r+1)\,q^{13r(3r-1)/2}$$

$$
\begin{aligned}
&\times f(-q^{26-39r},-q^{14+39r})f(-q^{806-39r},-q^{754+39r})\}\\
&-\{((-1)^r+1)\,(-q)^{13r(3r-1)/2}\\
&\times f(-(-q)^{26-39r},-(-q)^{14+39r})f(-(-q)^{806-39r},-(-q)^{754+39r})\}\Big]\\
=&\sum_{s=-5}^{4}\Big[\{(1+1)\,q^{26s(12s-1)}\\
&\times f(-q^{-156s+26},-q^{156s+14})f(-q^{-156s+806},-q^{156s+754})\}\\
&-\{(1+1)\,q^{26s(12s-1)}\\
&\times f(-q^{-156s+26},-q^{156s+14})f(-q^{-156s+806},-q^{156s+754})\}\Big]\\
=&\sum_{s=-5}^{4}0=0.
\end{aligned}
\tag{5.8.30}
$$

The proofs that $S_i=0$ for $i=1$ and $i=3$ are analogous.

Now we address S_2. Replace r by $4s+2$ in (5.8.28) in order to deduce that

$$
\begin{aligned}
S_2=&\sum_{\substack{-20\le r\le 19\\ r\equiv 2\,(\mathrm{mod}\,4)}}\Big[\{((-1)^r+1)\,q^{13r(3r-1)/2}\\
&\times f(-q^{26-39r},-q^{14+39r})f(-q^{806-39r},-q^{754+39r})\}\\
&-\{((-1)^r+1)\,(-q)^{13r(3r-1)/2}\\
&\times f(-(-q)^{26-39r},-(-q)^{14+39r})f(-(-q)^{806-39r},-(-q)^{754+39r})\}\Big]\\
=&\sum_{s=-5}^{4}\Big[2q^{13(2s+1)(12s+5)}f(-q^{-156s-52},-q^{156s+92})f(-q^{-156s+728},-q^{156s+832})\\
&-2(-1)q^{13(2s+1)(12s+5)}f(-q^{-156s-52},-q^{156s+92})f(-q^{-156s+728},-q^{156s+832})\Big]\\
=&\,4\sum_{s=-5}^{4}q^{13(2s+1)(12s+5)}f(-q^{-156s-52},-q^{156s+92})f(-q^{-156s+728},-q^{156s+832}).
\end{aligned}
\tag{5.8.31}
$$

Simplifying (5.8.31) with the help of (1.0.5), (1.0.4), and (5.8.12), we obtain

$$
\begin{aligned}
S_2=&\,4\Big(q^{6435}f(-q^{728},-q^{-688})f(-q^{1508},-q^{52})\\
&+q^{3913}f(-q^{572},-q^{-532})f(-q^{1352},-q^{208})\\
&+q^{2015}f(-q^{416},-q^{-376})f(-q^{1196},-q^{364})\\
&+q^{741}f(-q^{260},-q^{-220})f(-q^{1040},-q^{520})+q^{91}f(-q^{104},-q^{-64})f(-q^{884},-q^{676})\\
&+q^{65}f(-q^{-52},-q^{92})f(-q^{728},-q^{832})+q^{663}f(-q^{-208},-q^{248})f(-q^{572},-q^{988})\\
&+q^{1885}f(-q^{-364},-q^{404})f(-q^{416},-q^{1144})\\
&+q^{3731}f(-q^{-520},-q^{560})f(-q^{260},-q^{1300})\\
&+q^{6201}f(-q^{-676},-q^{716})f(-q^{104},-q^{1456})\Big)\\
=&\,4q^{171}f(-q^{8},-q^{32})f(-q^{52},-q^{1508})+4q^{105}f(-q^{12},-q^{28})f(-q^{208},-q^{1352})
\end{aligned}
$$

$$
\begin{aligned}
&+ 4q^{55} f(-q^{16}, -q^{24}) f(-q^{364}, -q^{1196}) + 4q^{21} f(-q^{20}, -q^{20}) f(-q^{520}, -q^{1040}) \\
&+ 4q^{3} f(-q^{16}, -q^{24}) f(-q^{676}, -q^{884}) + 4q f(-q^{12}, -q^{28}) f(-q^{728}, -q^{832}) \\
&+ 4q^{15} f(-q^{8}, -q^{32}) f(-q^{572}, -q^{988}) + 4q^{45} f(-q^{4}, -q^{36}) f(-q^{416}, -q^{1144}) \\
&+ 4q^{91} f(-q^{40}, -1) f(-q^{260}, -q^{1300}) - 4q^{149} f(-q^{4}, -q^{36}) f(-q^{104}, -q^{1456}) \\
= {}& 4q^{3} f(-q^{16}, -q^{24}) \{ f(-q^{676}, -q^{884}) + q^{52} f(-q^{364}, -q^{1196}) \} \\
&+ 4q^{15} f(-q^{8}, -q^{32}) \{ f(-q^{572}, -q^{988}) + q^{156} f(-q^{52}, -q^{1508}) \} \\
&+ 4q f(-q^{12}, -q^{28}) \{ f(-q^{728}, -q^{832}) + q^{104} f(-q^{208}, -q^{1352}) \} \\
&+ 4q^{45} f(-q^{4}, -q^{36}) \{ f(-q^{416}, -q^{1144}) - q^{104} f(-q^{104}, -q^{1456}) \} \\
&+ 4q^{21} f(-q^{20}, -q^{20}) f(-q^{520}, -q^{1040}). \qquad (5.8.32)
\end{aligned}
$$

By (1.0.4),

$$
f(-1, -q) f(q^{13}, q^{26}) = f(-1, q) f(-q^{13}, q^{26}) = 0. \qquad (5.8.33)
$$

Thus, utilizing (1.0.8), (1.0.3), (1.0.9), and (5.8.33), and then applying (5.8.27), (5.8.29), and (5.8.32), we conclude that

$$
\begin{aligned}
&\psi(q) f(-q^{13}) - \psi(-q) f(q^{13}) \\
&= \frac{1}{2} \bigg(f(1, q) f(-q^{13}, -q^{26}) - f(1, -q) f(q^{13}, -q^{26}) \\
&\quad + f(-1, -q) f(q^{13}, q^{26}) - f(-1, q) f(-q^{13}, q^{26}) \bigg) \\
&= \frac{1}{2} \sum_{i=0}^{3} S_i = \frac{1}{2} S_2 \\
&= 2q^{3} f(-q^{16}, -q^{24}) \{ f(-q^{676}, -q^{884}) + q^{52} f(-q^{364}, -q^{1196}) \} \\
&\quad + 2q^{15} f(-q^{8}, -q^{32}) \{ (-q^{572}, -q^{988}) + q^{156} f(-q^{52}, -q^{1508}) \} \\
&\quad + 2q f(-q^{12}, -q^{28}) \{ f(-q^{728}, -q^{832}) + q^{104} f(-q^{208}, -q^{1352}) \} \\
&\quad + 2q^{45} f(-q^{4}, -q^{36}) \{ f(-q^{416}, -q^{1144}) - q^{104} f(-q^{104}, -q^{1456}) \} \\
&\quad + 2q^{21} f(-q^{20}, -q^{20}) f(-q^{520}, -q^{1040}). \qquad (5.8.34)
\end{aligned}
$$

Now, by (5.8.12) with q replaced by q^8, and by (5.8.13) and (5.8.14), each with q replaced by q^{52},

$$
\begin{aligned}
&2q^{3} f(-q^{16}, -q^{24}) \{ f(-q^{676}, -q^{884}) + q^{52} f(-q^{364}, -q^{1196}) \} \\
&\quad + 2q^{15} f(-q^{8}, -q^{32}) \{ (-q^{572}, -q^{988}) + q^{156} f(-q^{52}, -q^{1508}) \} \\
&= \frac{2q^3}{\chi(-q^{52})} \left(g(q^8) g(q^{52}) + q^{12} h(q^8) h(q^{52}) \right). \qquad (5.8.35)
\end{aligned}
$$

By (1.0.7) and (1.0.9), we know that $\phi(-q^{20}) = f(-q^{20}, -q^{20})$ and $f(-q^{520}) = f(-q^{520}, -q^{1040})$. Therefore, applying (5.8.9) with q replaced by q^4, we deduce that

$$
\begin{aligned}
&2q f(-q^{12}, -q^{28}) \{ f(-q^{728}, -q^{832}) + q^{104} f(-q^{208}, -q^{1352}) \} \\
&\quad + 2q^{45} f(-q^{4}, -q^{36}) \{ f(-q^{416}, -q^{1144}) - q^{104} f(-q^{104}, -q^{1456}) \} \\
&\quad + 2q^{21} f(-q^{20}, -q^{20}) f(-q^{520}, -q^{1040}) \\
&= 2q f(-q^{8}) f(-q^{104}) \Big[G(q^{104}) H(-q^{4}) + q^{20} H(q^{104}) G(-q^{4}) \Big]
\end{aligned}
$$

$$\times \left[G(q^{16})G(q^{104}) + q^{24}H(q^{16})H(q^{104}) \right]. \tag{5.8.36}$$

Consequently, employing (5.8.35)–(5.8.36), we deduce from (5.8.34) that

$$\begin{aligned}
&\psi(q)f(-q^{13}) - \psi(-q)f(q^{13}) \\
&= \frac{2q^3}{\chi(-q^{52})} \left(g(q^8)g(q^{52}) + q^{12}h(q^8)h(q^{52}) \right) \\
&\quad + 2qf(-q^8)f(-q^{104}) \left[G(q^{104})H(-q^4) + q^{20}H(q^{104})G(-q^4) \right] \\
&\qquad \times \left[G(q^{16})G(q^{104}) + q^{24}H(q^{16})H(q^{104}) \right].
\end{aligned} \tag{5.8.37}$$

Next, we derive an analogous representation for

$$-qf(-q)\psi(q^{13}) - qf(q)\psi(-q^{13}).$$

Applying Theorem 1.0.3 with the parameters $\epsilon_1 = 0$, $\epsilon_2 = 1$, $a = 1$, $b = q^{13}$, $c = q$, $d = q^2$, $\alpha = 13$, $\beta = 3$ and $m = 40$, we deduce that

$$\begin{aligned}
&f(1, q^{13})f(-q, -q^2) \\
&= \sum_{r=-20}^{19} (-1)^r q^{r(3r-1)/2} f(-q^{260-39r}, -q^{260+39r}) f(-q^{80-3r}, -q^{40+3r}).
\end{aligned} \tag{5.8.38}$$

Applying Theorem 1.0.3 with the same set of parameters as for (5.8.38), except now with $\epsilon_1 = 1$ and $\epsilon_2 = 0$, we find that

$$\begin{aligned}
&f(-1, -q^{13})f(q, q^2) \\
&= \sum_{r=-20}^{19} q^{r(3r-1)/2} f(-q^{260-39r}, -q^{260+39r}) f(-q^{80-3r}, -q^{40+3r}).
\end{aligned} \tag{5.8.39}$$

Combining (5.8.38)–(5.8.39), along with the identities obtained by replacing q by $-q$ in each of them, we find that

$$\begin{aligned}
&f(1, q^{13})f(-q, -q^2) + f(1, -q^{13})f(q, -q^2) \\
&+ f(-1, -q^{13})f(q, q^2) + f(-1, q^{13})f(-q, q^2) = \sum_{i=0}^{3} S_i,
\end{aligned} \tag{5.8.40}$$

where

$$\begin{aligned}
S_i = \sum_{\substack{-20 \le r \le 19 \\ r \equiv i \pmod 4}} \Big[&\{ ((-1)^r + 1)\, q^{r(3r-1)/2} \\
&\times f(-q^{260-39r}, -q^{260+39r}) f(-q^{80-3r}, -q^{40+3r}) \} \\
&+ \{ ((-1)^r + 1)\, (-q)^{r(3r-1)/2} \\
&\times f(-(-q)^{260-39r}, -(-q)^{260+39r}) f(-(-q)^{80-3r}, -(-q)^{40+3r}) \} \Big].
\end{aligned}$$

It is easy to show directly that

$$S_1 = S_2 = S_3 = 0. \tag{5.8.41}$$

In S_0, replace r by $4s$, and simplify with the help of (1.0.5) and (1.0.4). Accordingly, we deduce that

$$
\begin{aligned}
S_0 = \sum_{\substack{-20\le r\le 19\\ r\equiv 0 \,(\mathrm{mod}\, 4)}} \Big[& \{((-1)^r+1)\, q^{r(3r-1)/2} \\
& \times f(-q^{260-39r},-q^{260+39r})f(-q^{80-3r},-q^{40+3r})\} \\
& + \{((-1)^r+1)\,(-q)^{r(3r-1)/2} \\
& \times f(-(-q)^{260-39r},-(-q)^{260+39r})f(-(-q)^{80-3r},-(-q)^{40+3r})\}\Big] \\
= 4\sum_{s=-5}^{4} & q^{2s(12s-1)} f(-q^{-156s+260},-q^{156s+260})f(-q^{-12s+80},-q^{12s+40}) \\
= 4\Big(& q^{610} f(-q^{1040},-q^{-520})f(-q^{140},-q^{-20}) \\
& + q^{392} f(-q^{884},-q^{-364})f(-q^{128},-q^{-8}) \\
& + q^{222} f(-q^{728},-q^{-208})f(-q^{116},-q^{4}) \\
& + q^{100} f(-q^{572},-q^{-52})f(-q^{104},-q^{16}) + q^{26} f(-q^{416},-q^{104})f(-q^{92},-q^{28}) \\
& + f(-q^{260},-q^{260})f(-q^{80},-q^{40}) + q^{22} f(-q^{104},-q^{416})f(-q^{68},-q^{52}) \\
& + q^{92} f(-q^{-52},-q^{572})f(-q^{56},-q^{64}) + q^{210} f(-q^{-208},-q^{728})f(-q^{44},-q^{76}) \\
& + q^{376} f(-q^{-364},-q^{884})f(-q^{32},-q^{88})\Big) \\
= 4q^{70} & f(-q^{520},-1)f(-q^{20},-q^{100}) + 4q^{20} f(-q^{364},-q^{156})f(-q^{8},-q^{112}) \\
& - 4q^{14} f(-q^{208},-q^{312})f(-q^{4},-q^{116}) - 4q^{48} f(-q^{52},-q^{468})f(-q^{16},-q^{104}) \\
& + 4q^{26} f(-q^{416},-q^{104})f(-q^{92},-q^{28}) + 4f(-q^{260},-q^{260})f(-q^{80},-q^{40}) \\
& + 4q^{22} f(-q^{104},-q^{416})f(-q^{68},-q^{52}) - 4q^{40} f(-q^{52},-q^{468})f(-q^{56},-q^{64}) \\
& - 4q^{2} f(-q^{208},-q^{312})f(-q^{44},-q^{76}) - 4q^{12} f(-q^{156},-q^{364})f(-q^{32},-q^{88}) \\
= 4q^{22} & f(-q^{104},-q^{416})\{f(-q^{68},-q^{52}) + q^{4} f(-q^{28},-q^{92})\} \\
& - 4q^{2} f(-q^{208},-q^{312})\{f(-q^{44},-q^{76}) + q^{12} f(-q^{4},-q^{116})\} \\
& - 4q^{12} f(-q^{156},-q^{364})\{f(-q^{32},-q^{88}) - q^{8} f(-q^{8},-q^{112})\} \\
& - 4q^{40} f(-q^{52},-q^{468})\{f(-q^{56},-q^{64}) + q^{8} f(-q^{16},-q^{104})\} \\
& + 4f(-q^{260},-q^{260})f(-q^{40},-q^{80}). \qquad (5.8.42)
\end{aligned}
$$

Observe that, by (1.0.4),

$$
f(-1,-q^{13})f(q,q^2) = f(-1,q^{13})f(-q,q^2) = 0. \qquad (5.8.43)
$$

Thus, utilizing (1.0.8), (1.0.3), (1.0.9) and (5.8.43), and then applying (5.8.40), (5.8.41), and (5.8.42), we conclude that

$$
\begin{aligned}
& - qf(-q)\psi(q^{13}) - qf(q)\psi(-q^{13}) \\
& = -\frac{q}{2}\Big(f(-q,-q^2)f(1,q^{13}) + f(q,-q^2)f(1,-q^{13}) \\
& \quad + f(-1,-q^{13})f(q,q^2) + f(-1,q^{13})f(-q,q^2)\Big)
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{q}{2}\sum_{i=0}^{3} S_i = -\frac{q}{2}S_0 \\
&= -2q^{23}f(-q^{104},-q^{416})\{f(-q^{68},-q^{52})+q^4f(-q^{28},-q^{92})\} \\
&\quad + 2q^3f(-q^{208},-q^{312})\{f(-q^{44},-q^{76})+q^{12}f(-q^4,-q^{116})\} \\
&\quad + 2q^{13}f(-q^{156},-q^{364})\{f(-q^{32},-q^{88})-q^8f(-q^8,-q^{112})\} \\
&\quad + 2q^{41}f(-q^{52},-q^{468})\{f(-q^{56},-q^{64})+q^8f(-q^{16},-q^{104})\} \\
&\quad - 2qf(-q^{260},-q^{260})f(-q^{40},-q^{80}). \qquad (5.8.44)
\end{aligned}
$$

Now, by (5.8.12) with q replaced by q^{104}, and by (5.8.13) and (5.8.14), each with q replaced by q^4,

$$
\begin{aligned}
&- 2q^{23}f(-q^{104},-q^{416})\{f(-q^{68},-q^{52})+q^4f(-q^{28},-q^{92})\} \\
&\quad + 2q^3f(-q^{208},-q^{312})\{f(-q^{44},-q^{76})+q^{12}f(-q^4,-q^{116})\} \\
&= \frac{2q^3}{\chi(-q^4)}\left(g(q^{104})h(q^4)-q^{20}h(q^{104})g(q^4)\right). \qquad (5.8.45)
\end{aligned}
$$

By (1.0.7) and (1.0.9), we know that $\phi(-q^{260}) = f(-q^{260},-q^{260})$ and $f(-q^{40}) = f(-q^{40},-q^{80})$. Hence, applying (5.8.10) with q replaced by q^4, we deduce that

$$
\begin{aligned}
&2q^{13}f(-q^{156},-q^{364})\{f(-q^{32},-q^{88})-q^8f(-q^8,-q^{112})\} \\
&\quad + 2q^{41}f(-q^{52},-q^{468})\{f(-q^{56},-q^{64})+q^8f(-q^{16},-q^{104})\} \\
&\quad - 2qf(-q^{260},-q^{260})f(-q^{40},-q^{80}) \\
&= -2q\{G(q^{208})H(q^8)-q^{40}G(q^8)H(q^{208})\} \\
&\quad \times \{G(q^8)G(-q^{52})-q^{12}H(q^8)H(-q^{52})\}f(-q^8)f(-q^{104}). \qquad (5.8.46)
\end{aligned}
$$

Consequently, employing (5.8.45)–(5.8.46), we deduce from (5.8.44) that

$$
\begin{aligned}
&-qf(-q)\psi(q^{13})-qf(q)\psi(-q^{13}) \\
&\quad = \frac{2q^3}{\chi(-q^4)}\left(g(q^{104})h(q^4)-q^{20}h(q^{104})g(q^4)\right) \\
&\qquad - 2qf(-q^8)f(-q^{104})\left[G(q^{208})H(q^8)-q^{40}G(q^8)H(q^{208})\right] \\
&\qquad\quad \times \left[G(q^8)G(-q^{52})-q^{12}H(q^8)H(-q^{52})\right]. \qquad (5.8.47)
\end{aligned}
$$

Hence, combining our representations for $\psi(q)f(-q^{13})-\psi(-q)f(q^{13})$ and $-qf(-q)\psi(q^{13})-qf(q)\psi(-q^{13})$, namely (5.8.37) and (5.8.47), we conclude that

$$
\begin{aligned}
&\frac{1}{4q^3}\left[\psi(q)f(-q^{13})-\psi(-q)f(q^{13})-qf(-q)\psi(q^{13})-qf(q)\psi(-q^{13})\right] \\
&\quad = \frac{1}{2\chi(-q^{52})}\left(g(q^8)g(q^{52})+q^{12}h(q^8)h(q^{52})\right) \\
&\qquad + \frac{1}{2q^2}f(-q^8)f(-q^{104})\left[G(q^{104})H(-q^4)+q^{20}H(q^{104})G(-q^4)\right] \\
&\qquad\quad \times \left[G(q^{16})G(q^{104})+q^{24}H(q^{16})H(q^{104})\right] \\
&\quad + \frac{1}{2\chi(-q^4)}\left(g(q^{104})h(q^4)-q^{20}h(q^{104})g(q^4)\right)
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{1}{2q^2} f(-q^8) f(-q^{104}) \Big[G(q^{208}) H(q^8) - q^{40} G(q^8) H(q^{208}) \Big] \\
&\qquad \times \Big[G(q^8) G(-q^{52}) - q^{12} H(q^8) H(-q^{52}) \Big]. \qquad (5.8.48)
\end{aligned}
$$

By (5.2.10), first with q replaced by $-q^4$, and second with q replaced by q^8, we deduce, respectively, that

$$
G(q^{104}) H(-q^4) + q^{20} H(q^{104}) G(-q^4) = G(q^8) G(-q^{52}) - q^{12} H(q^8) H(-q^{52}) \qquad (5.8.49)
$$

and

$$
G(q^{16}) G(q^{104}) + q^{24} H(q^{16}) H(q^{104}) = G(q^{208}) H(q^8) - q^{40} G(q^8) H(q^{208}). \qquad (5.8.50)
$$

Hence, the second and fourth terms on the right side of (5.8.48) cancel. Thus, we may rewrite (5.8.48) in the simpler form

$$
\begin{aligned}
&\frac{1}{4q^3} \left[\psi(q) f(-q^{13}) - \psi(-q) f(q^{13}) - q f(-q) \psi(q^{13}) - q f(q) \psi(-q^{13}) \right] \\
&= \frac{1}{2\chi(-q^{52})} \left(g(q^8) g(q^{52}) + q^{12} h(q^8) h(q^{52}) \right) \\
&\quad + \frac{1}{2\chi(-q^4)} \left(g(q^{104}) h(q^4) - q^{20} h(q^{104}) g(q^4) \right). \qquad (5.8.51)
\end{aligned}
$$

Employing (5.8.12) in (5.2.10), we find that

$$
\frac{g(q^2) g(q^{13})}{f(-q^2) f(-q^{13})} + q^3 \frac{h(q^2) h(q^{13})}{f(-q^2) f(-q^{13})} = \frac{g(q^{26}) h(q)}{f(-q) f(-q^{26})} - q^5 \frac{h(q^{26}) g(q)}{f(-q) f(-q^{26})}. \qquad (5.8.52)
$$

Therefore, multiplying (5.8.52) by $f(-q^2) f(-q^{26})$ and applying (1.0.12), we conclude that

$$
\frac{1}{\chi(-q^{13})} \left(g(q^2) g(q^{13}) + q^3 h(q^2) h(q^{13}) \right) = \frac{1}{\chi(-q)} \left(g(q^{26}) h(q) - q^5 h(q^{26}) g(q) \right),
$$

whence we deduce that

$$
\begin{aligned}
&\frac{1}{\chi(-q^{52})} \left(g(q^8) g(q^{52}) + q^{12} h(q^8) h(q^{52}) \right) \\
&= \frac{1}{\chi(-q^4)} \left(g(q^{104}) h(q^4) - q^{20} h(q^{104}) g(q^4) \right). \qquad (5.8.53)
\end{aligned}
$$

Employing (5.8.53) in (5.8.51), we complete the derivation of (5.8.24). This then completes the proof of a fascinating identity! □

Corollary 5.8.5.

$$
\begin{aligned}
&G(q^{104}) H(q^4) - q^{20} H(q^{104}) G(q^4) = G(q^8) G(q^{52}) + q^{12} H(q^8) H(q^{52}) \\
&= \sqrt{\frac{\chi(-q^{52})}{\chi(-q^4)} - q^4 \frac{\chi(-q^4)}{\chi(-q^{52})}}
\end{aligned}
$$

$$= \frac{1}{4q^3 f(-q^8) f(-q^{104})} \times \left[\psi(q) f(-q^{13}) - \psi(-q) f(q^{13}) - q\left(f(-q)\psi(q^{13}) + f(q)\psi(-q^{13})\right)\right]. \tag{5.8.54}$$

Proof. The first two equalities are immediate from (5.2.10) and (5.2.11). Next, divide (5.8.24) by $f(-q^4)f(-q^{104})$ to reveal that

$$\begin{aligned}
&\frac{1}{\chi(-q^4)}\left(\frac{g(q^{104})h(q^4)}{f(-q^4)f(-q^{104})} + q^{21}\frac{h(q^{104})h(q^4)}{f(-q^4)f(-q^{104})}\right) \\
&\quad = \frac{1}{4q^3 f(-q^4) f(-q^{104})}\Big[\psi(q) f(-q^{13}) - \psi(-q) f(q^{13}) \\
&\qquad\qquad - q\left(f(-q)\psi(q^{13}) + f(q)\psi(-q^{13})\right)\Big].
\end{aligned} \tag{5.8.55}$$

By (1.0.12),

$$\chi(-q^4) = \frac{f(-q^4)}{f(-q^8)}. \tag{5.8.56}$$

By (5.8.12),

$$\frac{g(q^{104})h(q^4)}{f(-q^4)f(-q^{104})} + q^{20}\frac{h(q^{104})h(q^4)}{f(-q^4)f(-q^{104})} = G(q^{104})H(q^4) + q^{21}H(q^{104})G(q^4). \tag{5.8.57}$$

Employing (5.8.56)–(5.8.57) in (5.8.55), we deduce that the first and final expressions in (5.8.54) are equal. This then completes the proof. □

We close this section with a curious corollary that provides alternative formulations for the expressions appearing in Lemma 5.8.3. Let S denote a subset of the rational numbers and $\sum_{n\in S} a_n q^n$ a generic q-series. Let t be a fixed rational number. We define an operator $\mathcal{L}_t$ acting on $\sum_{n\in S} a_n q^n$ by $\mathcal{L}_t(\sum_{n\in S} a_n q^n) = \sum_{n\in S'} a_n q^n$, where $S' \subseteq S$ is the set of all rational numbers in S that are congruent modulo 1 to t.

Corollary 5.8.6. *The following relations hold.*

$$\begin{aligned}
&\frac{1}{q^{8/5}}\mathcal{L}_{3/5}\left(\left[\phi(q^{-52/5}) + \phi(-q^{260})\right] f(-q^{8/5})\right) \\
&= -2\{G(q^{208})H(q^8) - q^{40}G(q^8)H(q^{208})\} \\
&\quad \times \{G(q^8)G(-q^{52}) - q^{12}H(q^8)H(-q^{52})\} f(-q^8) f(-q^{104}) \\
&= -2\{G(q^{104})H(-q^4) + q^{20}H(q^{104})G(-q^4)\} \\
&\quad \times \{G(q^{16})G(q^{104}) + q^{24}H(q^{16})H(q^{104})\} f(-q^8) f(-q^{104}) \\
&= \frac{1}{q^{4/5}}\mathcal{L}_{4/5}\left(\left[\phi(-q^{4/5}) + \phi(-q^{20})\right] f(-q^{104/5})\right).
\end{aligned} \tag{5.8.58}$$

Proof. In (2.5.24), add $2\phi(q^5)$ to both sides of the identity and replace q by $-q$.

We consequently find that

$$\phi(-q^{1/5}) + \phi(-q^5) = -2q^{1/5}f(-q^3, -q^7) + 2q^{4/5}f(-q, -q^9) + 2\phi(-q^5). \tag{5.8.59}$$

In (1.0.15), set $a = -q^{1/5}$, $b = -q^{2/5}$, and $n = 5$. After two applications of (1.0.5), we deduce that

$$\begin{aligned} f(-q^{1/5}) = f(-q^7, -q^8) &- q^{1/5}f(-q^5, -q^{10}) + qf(-q^2, -q^{13}) \\ &- q^{2/5}f(-q^4, -q^{11}) + q^{7/5}f(-q, -q^{14}). \end{aligned} \tag{5.8.60}$$

Replacing q by q^{52} in (5.8.59), we find that

$$\begin{aligned} &\phi(-q^{52/5}) + \phi(-q^{260}) \\ &\quad = -2q^{52/5}f(-q^{156}, -q^{364}) + 2q^{208/5}f(-q^{52}, -q^{468}) + 2\phi(-q^{260}). \end{aligned} \tag{5.8.61}$$

Replacing q by q^8 in (5.8.60) and also employing (1.0.9), we see that

$$\begin{aligned} f(-q^{8/5}) = f(-q^{56}, -q^{64}) &- q^{8/5}f(-q^{40}) + q^8f(-q^{16}, -q^{104}) \\ &- q^{16/5}f(-q^{32}, -q^{88}) + q^{56/5}f(-q^8, -q^{112}). \end{aligned} \tag{5.8.62}$$

Hence,

$$\begin{aligned} &\Big[\phi(q^{-52/5}) + \phi(-q^{260})\Big]\, f(-q^{8/5}) = -2q^{52/5}f(-q^{56}, -q^{64})f(-q^{156}, -q^{364}) \\ &\quad + 2q^{208/5}f(-q^{56}, -q^{64})f(-q^{52}, -q^{468}) + 2f(-q^{56}, -q^{64})\phi(-q^{260}) \\ &\quad + 2q^{12}f(-q^{40})f(-q^{156}, -q^{364}) - 2q^{216/5}f(-q^{40})f(-q^{52}, -q^{468}) \\ &\quad - 2q^{8/5}f(-q^{40})\phi(-q^{260}) - 2q^{92/5}f(-q^{16}, -q^{104})f(-q^{156}, -q^{364}) \\ &\quad + 2q^{248/5}f(-q^{16}, -q^{104})f(-q^{52}, -q^{468}) + 2q^8f(-q^{16}, -q^{104})\phi(-q^{260}) \\ &\quad + 2q^{68/5}f(-q^{32}, -q^{88})f(-q^{156}, -q^{364}) - 2q^{224/5}f(-q^{32}, -q^{88})f(-q^{52}, -q^{468}) \\ &\quad - 2q^{16/5}f(-q^{32}, -q^{88})\phi(-q^{260}) - 2q^{108/5}f(-q^8, -q^{112})f(-q^{156}, -q^{364}) \\ &\quad + 2q^{264/5}f(-q^8, -q^{112})f(-q^{52}, -q^{468}) + 2q^{56/5}f(-q^8, -q^{112})\phi(-q^{260}). \end{aligned} \tag{5.8.63}$$

From (5.8.63), we deduce that

$$\begin{aligned} &\frac{1}{2q^{8/5}}\mathcal{L}_{3/5}\left(\Big[\phi(q^{-52/5}) + \phi(-q^{260})\Big]\, f(-q^{8/5})\right) \\ &\quad = q^{40}f(-q^{56}, -q^{64})f(-q^{52}, -q^{468}) - f(-q^{40})\phi(-q^{260}) \\ &\quad + q^{48}f(-q^{16}, -q^{104})f(-q^{52}, -q^{468}) + q^{12}f(-q^{32}, -q^{88})f(-q^{156}, -q^{364}) \\ &\quad - q^{20}f(-q^8, -q^{112})f(-q^{156}, -q^{364}). \end{aligned} \tag{5.8.64}$$

Now, replace q by q^4 in (5.8.10), and observe that the right-hand side of the resulting equation matches the right-hand side of (5.8.64). Hence,

$$\begin{aligned} &\frac{1}{2q^{8/5}}\mathcal{L}_{3/5}\left(\Big[\phi(q^{-52/5}) + \phi(-q^{260})\Big]\, f(-q^{8/5})\right) \\ &\quad = -\{G(q^{208})H(q^8) - q^{40}G(q^8)H(q^{208})\} \\ &\qquad \times \{G(q^8)G(-q^{52}) - q^{12}H(q^8)H(-q^{52})\}f(-q^8)f(-q^{104}). \end{aligned} \tag{5.8.65}$$

This implies the first equality in (5.8.58).

Starting from (5.8.59) and (5.8.60) and proceeding analogously, but using (5.8.9) in place of (5.8.10), we derive that

$$\begin{aligned}
&\frac{1}{2q^{4/5}}\mathcal{L}_{4/5}\left(\left[\phi(-q^{4/5})+\phi(-q^{20})\right]f(-q^{104/5})\right)\\
&\quad=-\{G(q^{104})H(-q^4)+q^{20}H(q^{104})G(-q^4)\}\\
&\qquad\times\{G(q^{16})G(q^{104})+q^{24}H(q^{16})H(q^{104})\}f(-q^8)f(-q^{104}),
\end{aligned}\tag{5.8.66}$$

from which we deduce the third equality in (5.8.58).

Employing (5.8.49) and (5.8.50), we readily deduce that the expressions on the right-hand sides of (5.8.65) and (5.8.66) are equal. Therefore, the second equality in (5.8.58) holds. This completes the proof. □

Chapter 6

Further Modular Relations for the Göllnitz–Gordon Functions and Analogues

6.1 Further Relations for the Göllnitz–Gordon Functions

Recall the notation

$$f(-q^n) = f_n.$$

In his Ph.D. thesis [80], Robins utilized the theory of modular forms to discover and prove six relations for the Göllnitz–Gordon functions, namely,

$$S(q^3)T(q) + qS(q)T(q^3) = \frac{f_2 f_4 f_6^2 f_{24}}{f_1 f_3 f_8 f_{12}^2}, \tag{6.1.1}$$

$$S(q^3)T(q) - qS(q)T(q^3) = \frac{f_1 f_{12}}{f_3 f_4}, \tag{6.1.2}$$

$$S(q^7)T(q) - q^3 S(q)T(q^7) = 1, \tag{6.1.3}$$

$$S^2(q) + qT^2(q) = \frac{f_2^6}{f_1^3 f_4^3}, \tag{6.1.4}$$

$$S^2(q) - qT^2(q) = \frac{f_4^4}{f_1 f_2 f_8^2}, \tag{6.1.5}$$

$$\left(\frac{S(q)}{T(q)}\right)^2 + \left(\frac{qT(q)}{S(q)}\right)^2 = 6q + \frac{f_1^4 f_4^2}{f_2^2 f_8^4}. \tag{6.1.6}$$

Huang [57, 58], employing classical methods, in particular methods of Rogers and Bressoud, established 27 relations for the Göllnitz–Gordon functions that are in the spirit of the 40 relations given by Ramanujan for the Rogers – Ramanujan functions. Subsequently, Chen and Huang [35] expanded Huang's list of identities. Recently, Baruah, Bora, and Saikia [13] offered new proofs of many of the results of Chen and Huang; their methods yielded some additional identities as well.

Huang, in his thesis [57], discussed Robins' identities, and furthermore observed that his work included proofs of (6.1.2)–(6.1.5). He did not, however, give a proof of (6.1.1) or (6.1.6). In Section 3.6, we provided two new proofs of (6.1.1), as well as (6.1.2), employing methods different from those of Huang. We then applied (6.1.1) in Section 3.7 to prove a cubic modular relation for the Ramanujan–Göllnitz–Gordon continued fraction.

In this section, we prove the remaining identity not addressed by Huang, namely (6.1.6). We show that (6.1.6) may be derived from either (6.1.4) or (6.1.5). We then proceed to combine results from [58] and Section 5.7 in order to present three new relations for the Göllnitz–Gordon functions.

First Proof of (6.1.6). Recall from (3.2.24) that

$$S(q)T(q) = \frac{f_2 f_8^2}{f_1 f_4^2}. \tag{6.1.7}$$

Multiplying (6.1.6) by $S^2(q)T^2(q)$ and then applying (6.1.7) on the right-hand side of the resulting equality, we deduce that (6.1.6) is equivalent to

$$S^4(q) + q^2 T^4(q) = \left(6q + \frac{f_1^4 f_4^2}{f_2^2 f_8^4}\right) \frac{f_2^2 f_8^4}{f_1^2 f_4^4}. \tag{6.1.8}$$

Squaring (6.1.5) and using (6.1.7), we find that

$$S^4(q) + q^2 T^4(q) = \frac{f_4^8}{f_1^2 f_2^2 f_8^4} + 2q \frac{f_2^2 f_8^4}{f_1^2 f_4^4}. \tag{6.1.9}$$

Comparing (6.1.8) and (6.1.9), we see that, in order to prove (6.1.6), it suffices to show that

$$\left(6q + \frac{f_1^4 f_4^2}{f_2^2 f_8^4}\right) \frac{f_2^2 f_8^4}{f_1^2 f_4^4} = \frac{f_4^8}{f_1^2 f_2^2 f_8^4} + 2q \frac{f_2^2 f_8^4}{f_1^2 f_4^4}. \tag{6.1.10}$$

Multiplying (6.1.10) by

$$\left(\frac{f_2^2 f_8^4}{f_1^2 f_4^4}\right)^{-1}$$

and rearranging, we find that (6.1.10) is equivalent to

$$4q = \frac{f_4^{12}}{f_2^4 f_8^8} - \frac{f_1^4 f_4^2}{f_2^2 f_8^4}. \tag{6.1.11}$$

Applying Lemma 1.0.1, we write (6.1.11) in the equivalent form

$$4q = \frac{\psi^4(q^2)}{\psi^4(q^4)} - \frac{\phi^2(-q)}{\psi^2(q^4)}. \tag{6.1.12}$$

Hence, to complete our proof, we must establish (6.1.12). We present two proofs of (6.1.12).

First proof of (6.1.12). We utilize the representations [14, pp. 122–123, Entries 10(ii), 11(iii),(iv)], namely,

$$\phi(-q) = \sqrt{z}(1-\alpha)^{1/4},$$

$$\psi(q^2) = \frac{1}{2}\sqrt{z}\left(\alpha\frac{1}{q}\right)^{1/4}, \qquad \text{and} \qquad \psi(q^4) = \frac{1}{2}\sqrt{\frac{1}{2}z}\left((1-\sqrt{1-\alpha})\frac{1}{q}\right)^{1/2}.$$

Hence,

$$\frac{\psi^4(q^2)}{\psi^4(q^4)} - \frac{\phi^2(-q)}{\psi^2(q^4)} = \frac{\dfrac{1}{2^4}z^2\alpha\dfrac{1}{q}}{\dfrac{1}{2^4}\cdot\dfrac{1}{2^2}z^2\left(\left(1-\sqrt{1-\alpha}\right)\dfrac{1}{q}\right)^2} - \frac{z(1-\alpha)^{1/2}}{\dfrac{1}{2^2}\cdot\dfrac{1}{2}z\left(1-\sqrt{1-\alpha}\right)\dfrac{1}{q}}$$
$$= 4q\frac{\left(1-\sqrt{1-\alpha}\right)^2}{\left(1-\sqrt{1-\alpha}\right)^2} = 4q,$$

which is the required result. □

Second proof of (6.1.12). We require two identities from Ramanujan's notebooks, namely [14, p. 40, Entries 25(v),(vii)]

$$\phi^2(q) - \phi^2(-q) = 8q\psi^2(q^4), \tag{6.1.13}$$
$$\phi^4(q) - \phi^4(-q) = 16q\psi^4(q^2). \tag{6.1.14}$$

We begin with the trivial identity

$$\phi^2(q) - \phi^2(-q) = \left(\phi^2(q) + \phi^2(-q)\right) - 2\phi^2(-q).$$

Multiplying the preceding equation by

$$\frac{1}{16q}\left(\phi^2(q) - \phi^2(-q)\right), \tag{6.1.15}$$

we find that

$$4q\left(\frac{1}{8q}\left(\phi^2(q) - \phi^2(-q)\right)\right)^2$$
$$= \frac{1}{16q}\left(\phi^4(q) - \phi^4(-q)\right) - \phi^2(-q)\left(\frac{1}{8q}\left(\phi^2(q) - \phi^2(-q)\right)\right).$$

Applying (6.1.13)–(6.1.14) to the preceding equation, we conclude that

$$4q\psi^4(q^4) = \psi^4(q^2) - \phi^2(-q)\psi^2(q^4). \tag{6.1.16}$$

Dividing all terms in (6.1.16) by $\psi^4(q^4)$, we arrive at (6.1.12). This completes the proof. □

□

Second proof of (6.1.6). Beginning with (6.1.4), proceed as in the preceding proof. We thus determine that, in order to prove (6.1.6), it suffices to prove the theta function identity

$$8q = \frac{\phi^2(q)}{\psi^2(q^4)} - \frac{\phi^2(-q)}{\psi^2(q^4)}.$$

The preceding relation is trivially equivalent to (6.1.13), and so we are done. □

Next, we prove the following new relations as consequences of our work in

Section 5.7 and Huang's work in [58]. Define

$$A := \left[\frac{\chi(q^{12})\chi(-q^{48})}{\chi(q^4)\chi(-q^{16})} + q^3\frac{\chi(-q^{16})}{\chi(-q^{48})}\right].$$

Theorem 6.1.1. *We have*

(i) $$S(q^6)T(q^4) - qS(q^4)T(q^6) = \frac{\chi(-q)\chi(-q^8)}{\chi(-q^4)\chi(-q^6)\chi(-q^{24})\chi(-q^{48})} \times A,$$

(ii) $$S(q^{12})S(q^2) + q^7T(q^{12})T(q^2) = \frac{\chi(-q^3)}{\chi(-q^2)\chi(-q^4)\chi(-q^{16})} \times A,$$

(iii) $$\frac{S(q^6)T(q^4) - qS(q^4)T(q^6)}{S(q^{12})S(q^2) + q^7T(q^{12})T(q^2)} = \frac{\chi(-q)\chi(-q^2)\chi(-q^8)\chi(-q^{16})}{\chi(-q^3)\chi(-q^6)\chi(-q^{24})\chi(-q^{48})}.$$

Proof of (i). In [58, Equation (2.24)], Huang proved that

$$\frac{G(q^{96})H(q) - q^{19}G(q)H(q^{96})}{S(q^6)T(q^4) - qS(q^4)T(q^6)} = \frac{f(-q^4)f(-q^6)f(-q^{16})f(-q^{24})}{f(-q)f(-q^8)f(-q^{12})f(-q^{96})}. \tag{6.1.17}$$

Utilizing (6.1.17), (5.7.3), and (1.0.12), we thus see that

$$\begin{aligned}
&S(q^6)T(q^4) - qS(q^4)T(q^6)\\
&= \frac{f(-q)f(-q^8)f(-q^{12})f(-q^{96})}{f(-q^4)f(-q^6)f(-q^{16})f(-q^{24})}\left\{G(q^{96})H(q) - q^{19}G(q)H(q^{96})\right\}\\
&= \frac{f(-q)f(-q^8)f(-q^{12})f(-q^{96})}{f(-q^4)f(-q^6)f(-q^{16})f(-q^{24})} \cdot \frac{f(-q^8)}{f(-q^2)}\left\{q^3\frac{\chi(-q^{16})}{\chi(-q^{48})} + \frac{\chi(q^{12})\chi(-q^{48})}{\chi(q^4)\chi(-q^{16})}\right\}\\
&= \frac{f(-q)}{f(-q^2)} \cdot \frac{f(-q^8)}{f(-q^4)} \cdot \frac{f(-q^8)}{f(-q^{16})} \cdot \frac{f(-q^{12})}{f(-q^6)} \cdot \frac{f(-q^{48})}{f(-q^{24})} \cdot \frac{f(-q^{96})}{f(-q^{48})} \cdot A\\
&= \frac{\chi(-q)\chi(-q^8)}{\chi(-q^4)\chi(-q^6)\chi(-q^{24})\chi(-q^{48})} \times A. \qquad (6.1.18)
\end{aligned}$$

This completes the proof. □

Proof of (ii). By [58, Equation (2.25)], we know that

$$\frac{G(q^{32})G(q^3) + q^7H(q^{32})H(q^3)}{S(q^{12})S(q^2) + q^7T(q^{12})T(q^2)} = \frac{f(-q^2)f(-q^8)f(-q^{12})f(-q^{48})}{f(-q^3)f(-q^4)f(-q^{24})f(-q^{32})}. \tag{6.1.19}$$

Applying (6.1.19), (5.7.5), (1.0.12), and (1.0.13), and proceeding as in the proof of (i), we readily deduce the truth of (ii). □

Proof of (iii). Take the quotient of (i) and (ii). □

6.2 Modular Relations for Dodecic Analogues of the Rogers–Ramanujan Functions

The dodecic analogues of the Rogers–Ramanujan functions are:

$$X(q) := \sum_{n=0}^{\infty}\frac{(-q^2;q^2)_n(1-q^{n+1})q^{n(n+2)}}{(q;q)_{2n+2}} = \frac{(q;q^{12})_\infty(q^{11};q^{12})_\infty(q^{12};q^{12})_\infty}{(q;q)_\infty}, \tag{6.2.1}$$

$$Y(q) := \sum_{n=0}^{\infty} \frac{(-q^2;q^2)_{n-1}(1+q^n)q^{n^2}}{(q;q)_{2n}} = \frac{(q^5;q^{12})_\infty(q^7;q^{12})_\infty(q^{12};q^{12})_\infty}{(q;q)_\infty}. \tag{6.2.2}$$

The latter equality in each of the above identities is due to L.J. Slater [89]. Robins, using modular forms, found four modular identities for these dodecic analogues. Recently, Baruah and Bora [11] conducted a systematic study of modular relations for $X(q)$ and $Y(q)$, giving a multitude of new relations as well as applications to the theory of partitions.

In this section, we prove Robins' four relations for the dodecic analogues of the Rogers–Ramanujan functions, namely,

$$Y^2(q) - q^2X^2(q) = \frac{f_2f_6^5}{f_1^2f_3^2f_{12}^2}, \tag{6.2.3}$$

$$Y^2(q) + q^2X^2(q) = \frac{f_3^2f_4f_6}{f_1^2f_2f_{12}}, \tag{6.2.4}$$

$$X(q)Y(q^3) - q^2X(q^3)Y(q) = \frac{f_{18}^2}{f_3f_9}, \tag{6.2.5}$$

$$X(q)Y(q^3) + q^2X(q^3)Y(q) = \frac{f_4f_6^5f_9f_{36}}{f_2^2f_3^3f_{12}^2f_{18}} \tag{6.2.6}$$

Note that, by (1.0.6), (6.2.1) and (6.2.2) easily imply that

$$X(q) = \frac{f(-q,-q^{11})}{f(-q)}, \tag{6.2.7}$$

$$Y(q) = \frac{f(-q^5,-q^7)}{f(-q)}. \tag{6.2.8}$$

We give four proofs of (6.2.3) and three proofs of (6.2.4).

First Proof of (6.2.3) and (6.2.4). Use (6.2.7), (6.2.8), and (1.0.13) in order to rewrite (6.2.3) in the equivalent form

$$f^2(-q^5,-q^7) - q^2f^2(-q,-q^{11}) = f(-q^2)\phi(q^3) \tag{6.2.9}$$

and (6.2.4) in the equivalent form

$$f^2(-q^5,-q^7) + q^2f^2(-q,-q^{11}) = \frac{f^2(-q^3)f(-q^4)f(-q^6)}{f(-q^2)f(-q^{12})}. \tag{6.2.10}$$

Set $a = q^2$, $b = q^4$, and $c = d = -q^3$ in (1.0.16) and (1.0.17) in order to deduce, respectively, that,

$$f(q^2,q^4)f(-q^3,-q^3) + f(-q^2,-q^4)f(q^3,q^3) = 2f^2(-q^5,-q^7) \tag{6.2.11}$$

and

$$f(q^2,q^4)f(-q^3,-q^3) - f(-q^2,-q^4)f(q^3,q^3) = 2q^2f^2(-q,-q^{11}). \tag{6.2.12}$$

Subtracting (6.2.12) from (6.2.11) and employing (1.0.7) and (1.0.9), we deduce (6.2.9). This completes the proof of (6.2.3). Adding (6.2.11) and (6.2.12), we arrive at

$$f^2(-q^5,-q^7)+q^2f^2(-q,-q^{11})=f(q^2,q^4)f(-q^3,-q^3). \tag{6.2.13}$$

By (1.0.6) and (1.0.9), we find that

$$\begin{aligned}
f(q^2,q^4)f(-q^3,-q^3) &= (-q^2;q^6)_\infty(-q^4;q^6)_\infty(q^6;q^6)_\infty(q^3;q^6)^2_\infty(q^6;q^6)_\infty\\
&= \frac{(-q^2;q^2)_\infty}{(-q^6;q^6)_\infty}(q^6;q^6)_\infty\frac{(q^3;q^3)^2_\infty}{(q^6;q^6)^2_\infty}(q^6;q^6)_\infty\\
&= \frac{(q^4;q^4)_\infty(q^6;q^6)_\infty}{(q^2;q^2)_\infty(q^{12};q^{12})_\infty}(q^3;q^3)^2_\infty\\
&= \frac{f^2(-q^3)f(-q^4)f(-q^6)}{f(-q^2)f(-q^{12})}. \qquad (6.2.14)
\end{aligned}$$

Combining (6.2.14) with (6.2.13), we deduce (6.2.10), and thus complete the proof.

□

Second Proof of (6.2.3) and (6.2.4). We prove the equivalent forms (6.2.9) and (6.2.10).

Applying Theorem 1.0.3 with the parameters $a=q^2$, $b=q^4$, $c=d=q^3$, $m=4$, $\alpha=1$, $\beta=2$, $\epsilon_1=1$, $\epsilon_2=0$, we see that

$$\begin{aligned}
f(-q^2)\phi(q^3) &= f(-q^2,-q^4)f(q^3,q^3)\\
&= f(-q^5,-q^7)f(q^{26},q^{22})+q^3f(-q^{-1},-q^{13})f(q^{14},q^{34})\\
&\quad +q^{12}f(-q^{-7},-q^{19})f(q^2,q^{46})+q^{27}f(-q^{-13},-q^{25})f(q^{-10},q^{58})\\
&= f(-q^5,-q^7)f(q^{26},q^{22})-q^2f(-q^{11},-q)f(q^{14},q^{34})\\
&\quad -q^5f(-q^5,-q^7)f(q^2,q^{46})+q^3f(-q^{11},-q)f(q^{38},q^{10}), \qquad (6.2.15)
\end{aligned}$$

where we applied (1.0.5) several times in the last equality. By (1.0.15) with $a=-q^5$, $b=-q^7$, $n=2$; and with $a=-q$, $b=-q^{11}$, $n=2$, respectively,

$$\begin{aligned}
f(-q^5,-q^7) &= f(q^{22},q^{26})-q^5f(q^{46},q^2),\\
f(-q,-q^{11}) &= f(q^{14},q^{34})-qf(q^{38},q^{10}).
\end{aligned}$$

Hence,

$$\begin{aligned}
f^2(-q^5,-q^7)-q^2f^2(-q,-q^{11}) &= f(-q^5,-q^7)\left[f(q^{22},q^{26})-q^5f(q^{46},q^2)\right]\\
&\quad -q^2f(-q,-q^{11})\left[f(q^{14},q^{34})-qf(q^{38},q^{10})\right].
\end{aligned} \tag{6.2.16}$$

We easily see that the right-hand side of (6.2.16) matches the the right-hand side of (6.2.15). Thus, the left-hand sides of (6.2.16) and (6.2.15) also match. This completes the proof of (6.2.9).

To prove (6.2.10), use the equivalent form (6.2.13) and proceed analogously to the proof just given of (6.2.9). □

Third Proof of (6.2.3) and (6.2.4). We show that (6.2.9) and (6.2.10), and hence (6.2.3) and (6.2.4), may be derived from one another. We begin with the trivial identity

$$\left[f^2(-q^5,-q^7)+q^2f^2(-q,-q^{11})\right]^2=\left[f^2(-q^5,-q^7)-q^2f^2(-q,-q^{11})\right]^2 + 4q^2f^2(-q^5,-q^7)f^2(-q,-q^{11}). \quad (6.2.17)$$

Multiplying together (6.2.11) and (6.2.12), we find that

$$\begin{aligned}&4q^2f^2(-q^5,-q^7)f^2(-q,-q^{11})\\ &=f^2(q^2,q^4)f^2(-q^3,-q^3)-f^2(-q^2,-q^4)f^2(q^3,q^3).\end{aligned} \quad (6.2.18)$$

Employing (6.2.18) and the right-hand sides of (6.2.13) and (6.2.9), we find that

$$\left[f(q^2,q^4)\phi(-q^3)\right]^2=\left[f(-q^2,-q^4)\phi(q^3)\right]^2 + \left(f^2(q^2,q^4)\phi^2(-q^3)-f^2(-q^2,-q^4)\phi^2(q^3)\right). \quad (6.2.19)$$

The truth of (6.2.19) is immediate. From this it follows that (6.2.3) and (6.2.4) are indeed equivalent. □

Fourth Proof of (6.2.3). In [11, Equations (28),(29)], Baruah and Bora proved that

$$Y(q)+qX(q)=\frac{f^3(-q^2)}{f^2(-q)f(-q^4)} \quad (6.2.20)$$

and

$$Y(q)-qX(q)=\frac{f(-q^4)f^5(-q^6)}{f^2(-q^2)f^2(-q^3)f^2(-q^{12})}. \quad (6.2.21)$$

The truth of (6.2.3) is immediate upon multiplying (6.2.20) and (6.2.21). □

Next, we prove identities (6.2.5) and (6.2.6). We present two proofs for each of these relations.

First Proof of (6.2.5). Utilize (6.2.7), (6.2.8), and (1.0.12) in order to write (6.2.5) in the equivalent form

$$f(-q^{15},-q^{21})f(-q,-q^{11})-q^2f(-q^5,-q^7)f(-q^3,-q^{33})=\psi(q^9)f(-q). \quad (6.2.22)$$

Next, employ Theorem 1.0.3 with the set of parameters $a=1$, $b=q^9$, $c=q$, $d=q^2$, $\epsilon_1=0$, $\epsilon_2=1$, $\alpha=\beta=1$, $m=4$, to find that

$$\begin{aligned}f(1,q^9)f(-q,-q^2)&=f(-q^2,-q^{10})f(-q^{12},-q^{24})+f(-q,-q^{11})f(-q^{15},-q^{21})\\ &\quad-qf(-q^4,-q^8)f(-q^6,-q^{30})-q^2f(-q^5,-q^7)f(-q^3,-q^{33}),\end{aligned} \quad (6.2.23)$$

where we have used (1.0.5) to simplify the result. We employ Theorem 1.0.3 again with the same set of parameters, except now with $\epsilon_1=1$ and $\epsilon_2=0$, to find that

$$\begin{aligned}f(-1,-q^9)f(q,q^2)&=f(-q^2,-q^{10})f(-q^{12},-q^{24})-f(-q,-q^{11})f(-q^{15},-q^{21})\\ &\quad-qf(-q^4,-q^8)f(-q^6,-q^{30})+q^2f(-q^5,-q^7)f(-q^3,-q^{33}).\end{aligned} \quad (6.2.24)$$

By (1.0.4), the product on the left-hand side of (6.2.24) equals zero. Recalling the definitions (1.0.8) and (1.0.9), and utilizing (1.0.3), (6.2.23) and (6.2.24), we conclude that

$$\begin{aligned}\psi(q^9)f(-q) &= \frac{1}{2}f(1,q^9)f(-q,-q^2)\\ &= \frac{1}{2}\{f(1,q^9)f(-q,-q^2)-f(-1,-q^9)f(q,q^2)\}\\ &= f(-q,-q^{11})f(-q^{15},-q^{21})-q^2f(-q^5,-q^7)f(-q^3,-q^{33}).\end{aligned} \tag{6.2.25}$$

This last line is (6.2.22), which completes our proof. □

First Proof of (6.2.6). We first use (6.2.7), (6.2.8), and Lemma 1.0.1 in order to write (6.2.6) in the equivalent form

$$f(-q^{15},-q^{21})f(-q,-q^{11})+q^2f(-q^5,-q^7)f(-q^3,-q^{33})=\frac{1}{\chi(q)}\phi(q^3)\psi(-q^9). \tag{6.2.26}$$

Applying (1.0.6) and Lemma 1.0.1, we deduce that

$$\begin{aligned}f(q,q^2) &= (-q;q^3)_\infty(-q^2;q^3)_\infty(q^3;q^3)_\infty=\frac{(-q;q)_\infty}{(-q^3;q^3)_\infty}(q^3;q^3)_\infty\\ &= \frac{(q^2;q^2)_\infty(q^3;q^3)_\infty}{(q;q)_\infty(q^6;q^6)_\infty}(q^3;q^3)_\infty=\frac{f(-q^2)f^2(-q^3)}{f(-q)f(-q^6)}=\frac{\phi(-q^3)}{\chi(-q)}.\end{aligned} \tag{6.2.27}$$

Replacing q by $-q$ in (6.2.27), we arrive at

$$\frac{\phi(q^3)}{\chi(q)}=f(-q,q^2). \tag{6.2.28}$$

Using (6.2.28), we write (6.2.26) in the form

$$f(-q^{15},-q^{21})f(-q,-q^{11})+q^2f(-q^5,-q^7)f(-q^3,-q^{33})=f(-q,q^2)\psi(-q^9). \tag{6.2.29}$$

We now prove (6.2.29). Apply Theorem 1.0.3 with the set of parameters $a=1$, $b=-q^9$, $c=-q$, $d=q^2$, $m=8$, $\alpha=1$, $\beta=2$, $\epsilon_1=\epsilon_2=0$. Then apply Theorem 1.0.3 a second time with the same set of parameters, except now with $\epsilon_1=\epsilon_2=1$. Proceeding as in the proof of (6.2.6), we find that

$$\begin{aligned}f(-q,q^2)\psi(-q^9) &= \frac{1}{2}f(-q,q^2)f(1,-q^9)\\ &= \frac{1}{2}\{f(-q,q^2)f(1,-q^9)-f(q,-q^2)f(-1,q^9)\}\\ &= f(-q^{11},-q)f(q^{66},q^{78})-q^5f(-q^5,-q^7)f(q^{30},q^{114})\\ &\quad -q^{15}f(-q^{11},-q)f(q^{138},q^6)+q^2f(-q^5,-q^7)f(q^{102},q^{42})\\ &= f(-q^{11},-q)\left\{f(q^{66},q^{78})-q^{15}f(q^{138},q^6)\right\}\\ &\quad +q^2f(-q^5,-q^7)\left\{f(q^{102},q^{42})-q^3f(q^{30},q^{114})\right\}.\end{aligned} \tag{6.2.30}$$

By (1.0.15) with $a=-q^{15}$, $b=-q^{21}$, $n=2$, and with $a=-q^3$, $b=-q^{33}$,

$n = 2$, we obtain, respectively,

$$f(-q^{15}, -q^{21}) = f(q^{66}, q^{78}) - q^{15} f(q^{138}, q^6), \tag{6.2.31}$$

$$f(-q^3, -q^{33}) = f(q^{42}, q^{102}) - q^3 f(q^{114}, q^{30}). \tag{6.2.32}$$

Employing (6.2.31) and (6.2.32) in (6.2.30), we obtain (6.2.29). This completes the first proof of (6.2.6). □

Second Proof of (6.2.5) *and* (6.2.6). We show that (6.2.5) and (6.2.6) are equivalent. Thus, either one can be derived from the other. We work with the equivalent forms (6.2.22) and (6.2.26). We begin with the trivial identity

$$\begin{aligned}
&\left[f(-q^{15}, -q^{21}) f(-q, -q^{11}) - q^2 f(-q^5, -q^7) f(-q^3, -q^{33})\right]^2 \\
&\quad = \left[f(-q^{15}, -q^{21}) f(-q, -q^{11}) + q^2 f(-q^5, -q^7) f(-q^3, -q^{33})\right]^2 \\
&\quad\quad - 4q^2 f(-q^5, -q^7) f(-q, -q^{11}) f(-q^{15}, -q^{21}) f(-q^3, -q^{33}).
\end{aligned} \tag{6.2.33}$$

With the help of (1.0.6), it is not hard to show that

$$f(-q^5, -q^7) f(-q, -q^{11}) = \frac{f_1 f_6 f_{12}^2}{f_2 f_3}. \tag{6.2.34}$$

Substituting (6.2.34) and the right-hand sides of (6.2.9) and (6.2.26) into (6.2.33) and then using

$$f_1^2 = f^2(-q) = \frac{\phi^2(-q^2)}{\chi^2(q)}, \tag{6.2.35}$$

deducible from Lemma 1.0.1, we find that

$$\psi^2(q^9) \frac{\phi^2(-q^2)}{\chi^2(q)} = \psi^2(-q^9) \frac{\phi^2(q^3)}{\chi^2(q)} - 4q^2 \chi(-q) \chi(-q^3) \psi(q^3) \psi(q^6) \psi(q^9) \psi(q^{18}). \tag{6.2.36}$$

Thus, to prove the equivalence of (6.2.5) and (6.2.6), it is sufficient to prove the identity (6.2.36). To prove (6.2.36), we utilize the catalogue of theta function evaluations in [14, p. 122–124]. Let α, β, and γ be of the first, third, and ninth degrees, respectively. Let $m = z_1/z_3$ and $m' = z_3/z_9$ be the corresponding multipliers. Recall the evaluations [14, pp. 122–124, Entries 10(i),(iii); 11(i)-(iii); 12(v),(vi)]. Employing the indicated parameterizations in (6.2.36), dividing through by $z_3 z_9$, and then using $m = z_1/z_3$ and simplifying, we deduce that the theta function identity (6.2.36) is equivalent to

$$\begin{aligned}
m\gamma^{1/4}(1-\alpha)^{1/3}\alpha^{1/12} &= [\gamma(1-\gamma)]^{1/4} [\alpha(1-\alpha)]^{1/12} \\
&\quad - 2^{2/3}(1-\alpha)^{1/12}(1-\beta)^{1/12}\beta^{1/3}\gamma^{3/8}\alpha^{-1/24}.
\end{aligned} \tag{6.2.37}$$

We rewrite (6.2.37) as

$$m = \frac{[(1-\gamma)(1-\beta)]^{1/4}}{[(1-\alpha)(1-\beta)]^{1/4}} - 2^{2/3} \frac{[\beta(1-\beta)]^{1/3}}{[(1-\alpha)(1-\beta)]^{1/4}} \cdot \frac{\beta^{3/8}}{\alpha^{1/8}} \cdot \frac{\gamma^{1/8}}{\beta^{3/8}}. \tag{6.2.38}$$

Since α has degree three over β, and since γ has degree three over β, we deduce from (3.6.25) that

$$[(1-\alpha)(1-\beta)]^{1/4} = \frac{(m+1)(3-m)}{4m} \tag{6.2.39}$$

and

$$[(1-\gamma)(1-\beta)]^{1/4} = \frac{(m'+1)(3-m')}{4m'}. \tag{6.2.40}$$

From [14, p. 232, equation (5.1)], we also know that

$$\frac{\beta^{3/8}}{\alpha^{1/8}} = \frac{m-1}{2} \qquad \text{and} \qquad \frac{\gamma^{1/8}}{\beta^{3/8}} = \frac{2m'}{3+m'}. \tag{6.2.41}$$

The multipliers m and m' can be parameterized as rational functions of a parameter t. See [14, pp. 353–354] for details. We require the following parameterizations associated with the parameter t [14, p. 356, equation (3.15); p. 354, equations (3.10) and (3.11)]:

$$[\beta(1-\beta)]^{1/3} = \frac{2^{1/3}2t(1-t^3)}{m'^2}, \tag{6.2.42}$$

$$m^2 = \frac{(1+2t)^4}{1+8t^3}, \qquad \text{and} \qquad m'^2 = 1+8t^3. \tag{6.2.43}$$

Substituting (6.2.39)–(6.2.42) into (6.2.38), we find that

$$\begin{aligned} m = {} & \frac{(m'+1)(3-m')}{4m'} \cdot \frac{4m}{(m+1)(3-m)} \\ & - 2^{2/3}\left[2^{1/3}\frac{2t(1-t^3)}{m'^2}\right] \cdot \frac{4m}{(m+1)(3-m)} \cdot \frac{m-1}{2} \cdot \frac{2m'}{3+m'} \end{aligned} \tag{6.2.44}$$

Clearing denominators and canceling a common factor of m in (6.2.44), we arrive at

$$\begin{aligned} m'^2(m+1)(3-m)(3+m') = {} & m'(3+m')(m'+1)(3-m') \\ & - 16(m-1)m't(1-t^3) \end{aligned} \tag{6.2.45}$$

We now employ (6.2.43) repeatedly in (6.2.45), using positive square roots, in order to eliminate the multipliers m and m'. Expanding out the left- and right-hand sides of the resulting equation, we find that each side of (6.2.45) is equal to

$$(8+16t-8t^3-16t^4)\sqrt{1+8t^3} + (8-16t-8t^3+16t^4) - 64t^2 + 64t^5. \tag{6.2.46}$$

The truth of (6.2.38), and hence of (6.2.36), is now evident. This completes the proof of the equivalence of (6.2.5) and (6.2.6). □

Chapter 7

Sextodecic Relations

7.1 Introduction

In this chapter, we consider the following four sextodecic analogues of the Rogers-Ramanujan functions

$$I(q) := \frac{(q^7;q^{16})_\infty(q^9;q^{16})_\infty(q^{16};q^{16})_\infty}{(q;q)_\infty}, \tag{7.1.1}$$

$$J(q) := \frac{(q^5;q^{16})_\infty(q^{11};q^{16})_\infty(q^{16};q^{16})_\infty}{(q;q)_\infty}, \tag{7.1.2}$$

$$K(q) := \frac{(q^3;q^{16})_\infty(q^{13};q^{16})_\infty(q^{16};q^{16})_\infty}{(q;q)_\infty}, \tag{7.1.3}$$

$$L(q) := \frac{(q;q^{16})_\infty(q^{15};q^{16})_\infty(q^{16};q^{16})_\infty}{(q;q)_\infty}. \tag{7.1.4}$$

By applying various results on Ramanujan's theta functions, and methods of Bressoud [27], we find and establish several modular relations for $I(q)$, $J(q)$, $K(q)$, and $L(q)$. Some of these relations are connected with the Göllnitz–Gordon functions.

Invoking (1.0.6) in (7.1.1)–(7.1.4), we readily arrive at the following result.

Lemma 7.1.1. *We have*

$$I(q) = \frac{f(-q^7,-q^9)}{f(-q)}, \qquad J(q) = \frac{f(-q^5,-q^{11})}{f(-q)}, \tag{7.1.5}$$

$$K(q) = \frac{f(-q^3,-q^{13})}{f(-q)}, \qquad L(q) = \frac{f(-q,-q^{15})}{f(-q)}, \tag{7.1.6}$$

7.2 Modular Relations for $I(q)$, $J(q)$, $K(q)$, and $L(q)$

In this section, we present our list of modular relations involving some combinations of $I(q)$, $J(q)$, $K(q)$, and $L(q)$. For simplicity, for positive integers n, we set $I_n := I(q^n)$, $J_n := J(q^n)$, $K_n := K(q^n)$, $L_n := L(q^n)$.

$$I_1I_3 + qJ_1J_3 + q^6L_1L_3 + q^3K_1K_3 = \frac{f_2^2f_6^2}{f_1^2f_3f_{12}}, \tag{7.2.1}$$

$$I_1I_7 + q^2J_1J_7 + q^6K_1K_7 + q^{12}L_1L_7 = \frac{f_4^2f_7}{f_1f_2f_{14}}, \tag{7.2.2}$$

$$I_1I_{27}+q^7J_1J_{27}+q^{21}K_1K_{27}+q^{42}L_1L_{27}=\frac{f_9f_{12}}{f_1f_{27}}, \tag{7.2.3}$$

$$I_1I_{35}+q^9J_1J_{35}+q^{27}K_1K_{35}+q^{54}L_1L_{35}=\frac{f_7f_{20}}{f_1f_{35}}, \tag{7.2.4}$$

$$I_{11}J_1-q^4J_{11}L_1-q^8K_{11}I_1-q^{17}L_{11}K_1=\frac{f_4}{f_1}, \tag{7.2.5}$$

$$I_{11}L_1-q^2J_{11}K_1+q^7K_{11}J_1-q^{15}L_{11}I_1=\frac{f_{44}}{f_{11}}, \tag{7.2.6}$$

$$I_{39}K_1-q^9J_{39}I_1+q^{30}K_{39}L_1+q^{58}L_{39}J_1=\frac{f_3f_{12}f_{13}f_{52}}{f_1f_6f_{26}f_{39}}, \tag{7.2.7}$$

$$I_{10}L_2-qJ_{10}K_2+q^5K_{10}J_2-q^{12}I_2L_{10}=\frac{f_1f_{80}}{f_2f_{10}}, \tag{7.2.8}$$

$$I_{18}K_2-q^3J_{18}I_2+q^{15}K_{18}L_2+q^{26}L_{18}J_2=\frac{f_3f_{48}}{f_2f_{18}}, \tag{7.2.9}$$

$$I_{13}L_3-qJ_{13}K_3+q^6K_{13}J_3-q^{15}L_{13}I_3=\frac{f_1f_8f_{39}f_{156}}{f_3f_4f_{13}f_{78}}, \tag{7.2.10}$$

$$I_1I_{63}+q^{16}J_1J_{63}+q^{48}K_1K_{63}+q^{96}L_1L_{63}=\frac{f_7f_9f_{28}f_{36}}{f_{14}f_{18}}, \tag{7.2.11}$$

$$K_2J_6+q^6I_2L_6-q^3J_2K_6-L_2I_6=0. \tag{7.2.12}$$

The following identities are relations involving some combinations of $I(q)$, $J(q)$, $K(q)$, and $L(q)$, and the Göllnitz-Gordon functions $S(q)$ and $T(q)$.

$$\frac{K_1^2-qJ_1^2}{T(q)}=\frac{f_8^5}{f_1f_2f_4f_{16}^2}, \tag{7.2.13}$$

$$\frac{K_1^2+qJ_1^2}{T(-q)}=\frac{f_2}{f_1}, \tag{7.2.14}$$

$$\frac{I_1^2-q^3L_1^2}{S(q)}=\frac{f_8^5}{f_1f_2f_4f_{16}^2}, \tag{7.2.15}$$

$$\frac{I_1^2+q^3L_1^2}{S(-q)}=\frac{f_2}{f_1}. \tag{7.2.16}$$

7.3 Proof of (7.2.1)

Proof of (7.2.1). We apply Theorem 1.0.3 with the parameters $a=q$, $b=q^3$, $c=d=q^6$, $\alpha=1$, $\beta=3$, $\epsilon_1=0$, $\epsilon_2=1$, $m=4$. Using (1.0.7) and (1.0.8), we consequently find that

$$\begin{aligned}\phi(-q^6)\psi(q)&=f(-q^6,-q^6)f(q,q^3)\\&=f(-q^7,-q^9)f(-q^{27},-q^{21})-q^6f(-q^{-5},-q^{21})f(-q^{15},-q^{33})\\&\quad+q^{24}f(-q^{-17},-q^{33})f(-q^3,-q^{45})-q^{54}f(-q^{-29},-q^{45})f(-q^{-9},-q^{57})\\&=f(-q^7,-q^9)f(-q^{27},-q^{21})+qf(-q^{11},-q^5)f(-q^{15},-q^{33})\\&\quad+q^6f(-q^{15},-q)f(-q^3,-q^{45})+q^3f(-q^3,-q^{13})f(-q^{39},-q^9),\end{aligned} \tag{7.3.1}$$

where in the last equality we have applied (1.0.5) several times. Applying Lemma 1.0.1 and Lemma 7.1.1 to (7.3.1), we deduce (7.2.1). □

7.4 Proofs of (7.2.2)–(7.2.12)

We apply the method given by Bressoud in his thesis [27]. Here, we use f_n instead of P_n, and the variable q instead of x which stands for q^2 in [27]. The letters α, β, m, n, p always denote positive integers, and m must be odd. Following Bressoud [27], we define

$$\underline{g}_{\alpha}^{(p,n)} = \{q^{(12n^2-12n+3-p(p-\frac{1}{2}))\alpha/(12p)}\}\frac{(q^{(p+1-2n)\alpha};q^{2p\alpha})_\infty(q^{(p-1+2n)\alpha};q^{2p\alpha})_\infty}{(q^{\alpha};q^{2\alpha})_\infty}. \tag{7.4.1}$$

Proposition 7.4.1. [27, Proposition 5.8]

$$\underline{g}_{\alpha}^{(p,n)} = \underline{g}_{\alpha}^{(p,-n+1)}, \quad \text{and} \quad \underline{g}_{\alpha}^{(p,n)} = -\underline{g}_{\alpha}^{(p,n-p)}, \quad \text{and} \quad \underline{g}_{\alpha}^{(p,n)} = -\underline{g}_{\alpha}^{(p,p-n+1)}. \tag{7.4.2}$$

Proposition 7.4.2. [27, Proposition 5.9]

$$\underline{g}_{\alpha}^{(2,1)} = 1. \tag{7.4.3}$$

Proposition 7.4.3. [57, Proposition 4.2]

$$\underline{g}_{\alpha}^{(4,1)} = q^{-11/48}\frac{f_{4\alpha}}{f_{8\alpha}}S(q^{\alpha}) \tag{7.4.4}$$

$$\underline{g}_{\alpha}^{(4,2)} = q^{13/48}\frac{f_{4\alpha}}{f_{8\alpha}}T(q^{\alpha}). \tag{7.4.5}$$

Proposition 7.4.4.

$$\underline{g}_{\alpha}^{(8,1)} = q^{-57\alpha/96}\frac{f_{2\alpha}}{f_{16\alpha}}I_{\alpha}, \tag{7.4.6}$$

$$\underline{g}_{\alpha}^{(8,2)} = q^{-33\alpha/96}\frac{f_{2\alpha}}{f_{16\alpha}}J_{\alpha}, \tag{7.4.7}$$

$$\underline{g}_{\alpha}^{(8,3)} = q^{15\alpha/96}\frac{f_{2\alpha}}{f_{16\alpha}}K_{\alpha}, \tag{7.4.8}$$

$$\underline{g}_{\alpha}^{(8,4)} = q^{87\alpha/96}\frac{f_{2\alpha}}{f_{16\alpha}}L_{\alpha}. \tag{7.4.9}$$

where $I(q)$, $J(q)$, $K(q)$, and $L(q)$ are as defined in (7.1.1)–(7.1.4)*.*

Proof. Take $p = 8$ and $n = 1$ in (7.4.1). Then

$$\underline{g}_{\alpha}^{(8,1)} = q^{-57\alpha/96}\frac{(q^{7\alpha};q^{16\alpha})_\infty(q^{9\alpha};q^{16\alpha})_\infty(q^{16\alpha};q^{16\alpha})_\infty}{(q^{\alpha};q^{2\alpha})_\infty(q^{16\alpha};q^{16\alpha})_\infty}. \tag{7.4.10}$$

Employing (1.0.6) and Lemma 7.1.1 in (7.4.10), we readily deduce (7.4.6). Similarly, we can prove (7.4.7)–(7.4.9). □

Theorem 7.4.5. [27, Proposition 5.4] *For odd $p > 1$,*

$$\phi_{\alpha,\beta,m,p} = 2q^{(\alpha+\beta)/12}f_{2\alpha}f_{2\beta}\sum_{n=1}^{(p-1)/2}\left(g_{\beta}^{(p,n)}g_{\alpha}^{(p,mn-\frac{m-1}{2})}\right) \tag{7.4.11}$$

Lemma 7.4.6. [27, Corollary 5.5 and 5.6] *If $\phi_{\alpha,\beta,m,p}$ is defined by (7.4.11), then*

$$\phi_{\alpha,\beta,m,1} = 0, \tag{7.4.12}$$

$$\phi_{\alpha,\beta,1,3} = 2q^{(\alpha+\beta)/24} f_\alpha f_\beta. \tag{7.4.13}$$

Theorem 7.4.7. [27, Proposition 5.10] *For even p,*

$$\phi_{\alpha,\beta,m,p} = 2q^{(2p-1)(\alpha+\beta)/24} \frac{f_{2p\alpha} f_{2p\beta} f_\alpha f_\beta}{f_{2\alpha} f_{2\beta}} \sum_{n=1}^{p/2} \left(g_\beta^{(p,n)} g_\alpha^{(p,mn-\frac{m-1}{2})} \right) \tag{7.4.14}$$

Lemma 7.4.8. [27, Corollary 5.11] *If α and β are even, positive integers, then*

$$\phi_{\alpha,\beta,1,2} = 2q^{(\alpha+\beta)/16} \frac{f_{2\alpha} f_{2\beta} f_{\alpha/2} f_{\beta/2}}{f_\alpha f_\beta}. \tag{7.4.15}$$

Proposition 7.4.9. [51, Lemma 5.1]

$$\begin{aligned}&\phi_{\alpha,\beta,1,4}\\ &= 2q^{(\alpha+\beta)/32}\{S(q^{\beta/2})S(q^{\alpha/2}) + q^{(\alpha+\beta)/4}T(q^{\beta/2})T(q^{\alpha/2})\} \frac{f_{2\alpha} f_{2\beta} f_{\alpha/2} f_{\beta/2}}{f_\alpha f_\beta}.\end{aligned} \tag{7.4.16}$$

$$\begin{aligned}&\phi_{\alpha,\beta,3,4}\\ &= 2q^{(9\alpha+\beta)/32}\{S(q^{\beta/2})T(q^{\alpha/2}) - q^{(\beta-\alpha)/4}S(q^{\alpha/2})T(q^{\beta/2})\} \frac{f_{2\alpha} f_{2\beta} f_{\alpha/2} f_{\beta/2}}{f_\alpha f_\beta}.\end{aligned} \tag{7.4.17}$$

Proposition 7.4.10.

$$\begin{aligned}&\phi_{\alpha,\beta,1,8}\\ &= 2q^{(\alpha+\beta)/32} f_\alpha f_\beta \{I_\alpha I_\beta + q^{(\alpha+\beta)/4} J_\alpha J_\beta + q^{3(\alpha+\beta)/4} K_\alpha K_\beta + q^{3(\alpha+\beta)/2} L_\alpha L_\beta\}.\end{aligned} \tag{7.4.18}$$

$$\begin{aligned}&\phi_{\alpha,\beta,3,8}\\ &= 2q^{(\alpha+\beta)/32} f_\alpha f_\beta \{q^{\alpha/4} I_\beta J_\alpha - q^{\frac{3}{2}\alpha+\frac{\beta}{4}} J_\beta L_\alpha - q^{\frac{3}{4}\beta} K_\beta I_\alpha - q^{\frac{3}{4}\alpha+\frac{3}{2}\beta} L_\beta K_\alpha\}.\end{aligned} \tag{7.4.19}$$

$$\begin{aligned}&\phi_{\alpha,\beta,5,8}\\ &= 2q^{(\alpha+\beta)/32} f_\alpha f_\beta \{q^{\frac{3}{4}\alpha} I_\beta K_\alpha - q^{\frac{\beta}{4}} J_\beta I_\alpha + q^{\frac{3}{2}\alpha+\frac{3}{4}\beta} K_\beta L_\alpha + q^{\frac{\alpha}{4}+\frac{3}{2}\beta} L_\beta J_\alpha\}.\end{aligned} \tag{7.4.20}$$

$$\begin{aligned}&\phi_{\alpha,\beta,7,8}\\ &= 2q^{(\alpha+\beta)/32} f_\alpha f_\beta \{q^{\frac{3}{2}\alpha} I_\beta L_\alpha - q^{\frac{3}{4}\alpha+\frac{\beta}{4}} J_\beta K_\alpha + q^{\frac{\alpha}{4}+\frac{3}{4}\beta} K_\beta J_\alpha - q^{\frac{3}{2}\beta} L_\beta I_\alpha\}.\end{aligned} \tag{7.4.21}$$

Proof. Apply equation (7.4.14) with $m = 7$ and $p = 8$ to obtain

$$\begin{aligned}\phi_{\alpha,\beta,7,8} &= 2q^{15(\alpha+\beta)/24}\\ &\times \frac{f_{16\alpha} f_{16\beta} f_\alpha f_\beta}{f_{2\alpha} f_{2\beta}} \{\underline{g}_\beta^{(8,1)} \underline{g}_\alpha^{(8,4)} + \underline{g}_\beta^{(8,2)} \underline{g}_\alpha^{(8,11)} + \underline{g}_\beta^{(8,3)} \underline{g}_\alpha^{(8,18)} + \underline{g}_\beta^{(8,4)} \underline{g}_\alpha^{(8,25)}\}.\end{aligned} \tag{7.4.22}$$

With the help of Proposition 7.4.1 and equations (7.4.6)–(7.4.9), we readily deduce (7.4.21). The proofs of (7.4.18)–(7.4.19) are similar; we omit the details. □

Theorem 7.4.11. [27, Corollary 7.3] *Let α_i, β_i, m_i, p_i be positive integers for $i = 1, 2$. In addition, m_1 and m_2 must be odd. Let $\lambda_1 := (\alpha_1 m_1^2 + \beta_1)/p_1$ and $\lambda_2 := (\alpha_2 m_2^2 + \beta_2)/p_2$. If the conditions*

$$\begin{aligned} \lambda_1 &= \lambda_2, \\ \alpha_1\beta_1 &= \alpha_2\beta_2, \\ \alpha_1 m_1 &\equiv \pm\alpha_2 m_2 \pmod{\lambda_1} \end{aligned}$$

all hold, then

$$\phi_{\alpha_1,\beta_1,m_1,p_1} = \phi_{\alpha_2,\beta_2.m_2,p_2}. \tag{7.4.23}$$

Proposition 7.4.12. [58, Proposition 5.4] *For $p > 1$,*

$$\phi_{1,p-1,1,p} = q^{1/4} f(1, q^2) f(-q^{p-1}, -q^{p-1}). \tag{7.4.24}$$

Furthermore, identity (7.2.2) *holds.*

Proof. Set $p = 8$ in (7.4.24) and apply (1.0.6) to find that

$$\begin{aligned} \phi_{1,7,1,8} &= q^{1/4} f(1, q^2) f(-q^7, -q^7) \\ &= q^{1/4}(-1; q^2)_\infty(-q^2; q^2)_\infty(q^2; q^2)_\infty(q^7; q^{14})^2_\infty(q^{14}; q^{14})_\infty \\ &= 2q^{1/4}\left(\frac{(q^4; q^4)_\infty}{(q^2; q^2)_\infty}\right)^2 (q^2; q^2)_\infty \left(\frac{(q^7; q^7)_\infty}{(q^{14}; q^{14})_\infty}\right)^2 (q^{14}; q^{14})_\infty \\ &= 2q^{1/4}\left(\frac{f_4}{f_2}\right)^2 f_2 \left(\frac{f_7}{f_{14}}\right)^2 f_{14} \\ &= 2q^{1/4}\frac{f_4^2 f_7^2}{f_2 f_{14}}. \end{aligned} \tag{7.4.25}$$

Apply (7.4.18) with $p = 8$ to find that

$$\phi_{1,7,1,8} = 2q^{8/32} f_1 f_7 \{I_1 I_7 + q^2 J_1 J_7 + q^6 K_1 K_7 + q^{12} L_1 L_7\}. \tag{7.4.26}$$

Combining (7.4.25) and (7.4.26), we obtain (7.2.2). □

Let $\mathbf{N}$ denote the natural numbers.

Proposition 7.4.13. [51, Proposition 6.13] *For $p \in \mathbf{N}$,*

$$\phi_{2,p(p+3),1,p+2} = \phi_{p+3,2p,1,3}. \tag{7.4.27}$$

Furthermore, identity (7.2.3) holds.

Proof. Set $p = 6$ in (7.4.27) to obtain (7.2.3) with the help of (7.4.18) and (7.4.13). □

Proposition 7.4.14. [51, Proposition 6.19] *For $p \in \mathbf{N}$,*

$$\phi_{2,p^2+3p,1,p+1} = \phi_{2p+6,p,1,3}. \tag{7.4.28}$$

Furthermore, identity (7.2.4) holds.

Proof. Setting $p = 7$ in (7.4.28), we obtain (7.2.3) by using (7.4.18) and (7.4.13). $\square$

Proposition 7.4.15. [58, Proposition 6.1] *For $p \in \mathbf{N}$, p even,*

$$\phi_{2,6p+10,p+1,2p+4} = \phi_{4,3p+5,1,3}. \tag{7.4.29}$$

Furthermore, identity (7.2.5) holds.

Proof. Set $p = 2$ in (7.4.29), and then employ (7.4.19) and (7.4.13). Replacing q^2 by q in the resulting equation, we readily obtain (7.2.5). $\square$

Proposition 7.4.16. [58, Proposition 6.2] *For $p \in \mathbf{N}$, p even,*

$$\phi_{2,3p+10,p+3,p+4} = \phi_{1,6p+20,1,3}. \tag{7.4.30}$$

Furthermore, identity (7.2.6) holds.

Proof. We set $p = 4$ in (7.4.30) to obtain (7.2.6) with the help of (7.4.21) and (7.4.13). $\square$

Proposition 7.4.17. [51, Proposition 6.15] *For $p \in \mathbf{N}$,*

$$\phi_{1,p^2+10p,5,p+5} = \phi_{p+10,p,1,2}. \tag{7.4.31}$$

Furthermore, identity (7.2.7) holds.

Proof. Set $p = 3$ in (7.4.31) and utilize (7.4.20) and (7.4.15) to deduce (7.2.7). $\square$

Proposition 7.4.18. [58, Proposition 6.3] *For $p \in \mathbf{N}$, p even,*

$$\phi_{4,3p+8,p+3,p+4} = \phi_{1,12p+3,1,3}. \tag{7.4.32}$$

Furthermore, identity (7.2.8) holds.

Proof. Setting $p = 4$ in (7.4.32) and applying (7.4.21) and (7.4.13), we readily obtain (7.2.8). $\square$

Proposition 7.4.19. [51, Proposition 6.23] *For $p \in \mathbf{N}$,*

$$\phi_{p+1,4p^2,5,p+5} = \phi_{p,4p(p+1),1,p}. \tag{7.4.33}$$

Furthermore, identity (7.2.9) holds.

Proof. Set $p = 3$ in (7.4.33) and utilize (7.4.20) and (7.4.13) to obtain (7.2.9). $\square$

Proposition 7.4.20. [58, Proposition 6.3] *For $p \in \mathbf{N}$, p even,*

$$\phi_{6,4p+10,p+3,p+4} = \phi_{2,12p+30,1,2}. \tag{7.4.34}$$

Furthermore, identity (7.2.10) holds.

Proof. Set $p = 4$ in (7.4.34). Employing (7.4.21) and (7.4.15), we readily obtain (7.2.10). $\square$

Proposition 7.4.21. *For $p \in \mathbf{N}$,*

$$\phi_{p^2+12p+35,1,1,p+6} = \phi_{p+7,p+5,1,2}. \tag{7.4.35}$$

Furthermore, identity (7.2.11) holds.

Proof. Let

$$
\begin{array}{llll}
\alpha_1 = p^2 + 12p + 35, & \beta_1 = 1, & m_1 = 1, & p_1 = p + 6 \\
\alpha_2 = p + 7, & \beta_2 = p + 5, & m_2 = 1, & p_2 = 2.
\end{array}
$$

Then the hypotheses of Theorem 7.4.11 are satisfied with $\lambda_1 = \lambda_2 = p + 6$. Hence, by Theorem 7.4.11, (7.4.35) holds. In particular, $p = 2$ implies

$$\phi_{63,1,1,8} = \phi_{9,7,1,2}. \tag{7.4.36}$$

Applying (7.4.18) and (7.4.15), and noting that both expressions are symmetric in α and β, we readily obtain (7.2.11). □

Proposition 7.4.22.

$$\phi_{12p,4p,7,8} = \phi_{2p,24p,5,1}. \tag{7.4.37}$$

Furthermore, identity (7.2.11) holds.

Proof. Equality (7.4.37) holds by Theorem 7.4.11 with $\lambda_1 = \lambda_2 = 74p$. Moreover, setting, $p = 1$ in (7.4.37) and utilizing (7.4.21) and (7.4.12), we deduce (7.2.11). □

7.5 Proofs of (7.2.13)–(7.2.16)

Proposition 7.5.1. *Identity 7.2.13 holds.*

Proof. Set $a = q$, $b = q^7$, $c = d = q^4$ in (1.0.16) and (1.0.17). Add the resulting equations and then replace q by $-q$ to obtain

$$f(-q, -q^7)f(q^4, q^4) = f^2(-q^5, -q^{11}) - qf^2(-q^3, -q^{13}). \tag{7.5.1}$$

Using (1.0.7) we rewrite (7.5.1) in the equivalent form

$$\frac{f_1 f_4}{f_2}\left(\frac{f_2}{f_1 f_4} f(-q, -q^7)\right)\phi(q^4) = f_1^2\left(\frac{f^2(-q^5, -q^{11})}{f_1^2} - q\frac{f^2(-q^3, -q^{13})}{f_1^2}\right). \tag{7.5.2}$$

Using (3.2.23) and Lemma 1.0.1 in (7.5.2), we obtain (7.2.13). □

Proposition 7.5.2. *Identity (7.2.14) holds.*

Proof. With the same set of parameters as in the proof of (7.2.13), subtract (1.0.17) from (1.0.16), and replace q by $-q$ in the difference to obtain

$$f(q, q^7)f(-q^4, -q^4) = f^2(-q^5, -q^{11}) + qf^2(-q^3, -q^{13}). \tag{7.5.3}$$

Using (1.0.7) we rewrite (7.5.1) in the equivalent form

$$\frac{f(q)f_4}{f_2}\left(\frac{f_2}{f(q)f_4} f(q, q^7)\right)\phi(-q^4) = f_1^2\left(\frac{f^2(-q^5, -q^{11})}{f_1^2} + q\frac{f^2(-q^3, -q^{11})}{f_1^2}\right). \tag{7.5.4}$$

Employing (3.2.23) and Lemma 1.0.1, we deduce (7.2.14). □

Proposition 7.5.3. *Identities (7.2.15) and (7.2.16) hold.*

Proof. With the set of parameters $a = q^3$, $b = q^5$, $c = d = q^4$, we can by similar methods to the proofs of (7.2.13) and (7.2.14) prove (7.2.15) and (7.2.16). $\square$

Chapter 8

An Elementary Proof of a Theta Function Transformation Formula

With $q = e^{2\pi i z}$, $\Im(z) > 0$, we define

$$\Theta_2(z) := 2q^{1/4} \prod_{n\geq 1} \left(\frac{1-q^{4n}}{1-q^{4n-2}} \right) = 2 \sum_{n\geq 0} q^{(2n+1)^2/4} \tag{8.0.1}$$

and

$$\Theta_4(z) := \prod_{n\geq 1} \left(\frac{1-q^n}{1+q^n} \right) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2}. \tag{8.0.2}$$

These theta functions satisfy the transformation formula

$$\Theta_4 \left(\frac{1}{-z} \right) = \sqrt{-iz}\,\Theta_2(z). \tag{8.0.3}$$

We give an elementary proof of the logarithmic form of (8.0.3), namely (8.0.4), following the method of Berndt-Venkatachaliengar [22]. In [64], S. Kongsiriwong adapted the ideas in [22] to prove the analogous inversion formula for $\Theta_3(z)$. These ideas have subsequently been extended by Berndt, Gugg, Kongsiriwong, and J. Thiel [20] to yield an elementary proof of a far more general transformation formula, which includes as special cases the formulas proved in [22], [64], as well as (8.0.3).

Theorem 8.0.4. *If $\Im(z) > 0$, then*

$$\log \Theta_4 \left(\frac{-1}{z} \right) = -\frac{\pi}{4} i + \frac{1}{2} \log z + \log \Theta_2(z), \tag{8.0.4}$$

where $\log z$ *is the principal branch with* $-\pi < arg(z) < \pi$.

Proof. For any complex number z with $\Im(z) > 0$, setting $q = e^{\pi i z}$, we find that

$$\begin{aligned} \log \Theta_4(z) &= \sum_{n\geq 1} \left(\log(1-q^n) - \log(1+q^n) \right) \\ &= \sum_{n\geq 1} \left(\sum_{m\geq 1} \frac{-1}{m} q^{nm} + \sum_{m\geq 1} \frac{1}{m} (-q^n)^m \right) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{m\geq 1}\left(\frac{-1}{m}\sum_{n\geq 1}(q^m)^n + \frac{(-1)^m}{m}\sum_{n\geq 1}(q^m)^n\right)\\
&= \sum_{m\geq 1}\left(\frac{-1}{m}\left(\frac{q^m}{1-q^m}\right) + \frac{(-1)^m}{m}\left(\frac{q^m}{1-q^m}\right)\right)\\
&= \sum_{m\geq 1}\left(\frac{-1}{m}\left(\frac{1}{q^{-m}-1}\right) + \frac{(-1)^m}{m}\left(\frac{1}{q^{-m}-1}\right)\right)\\
&= \sum_{m\geq 1}\frac{-2}{2m-1}\left(\frac{1}{q^{-(2m-1)}-1}\right)\\
&= \sum_{m\geq 1}\frac{-2}{(2m-1)\left(e^{-(2m-1)\pi i z}-1\right)}. \qquad (8.0.5)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\log \Theta_2(z) &= \log 2 + \frac{1}{4}\log q + \sum_{n\geq 1}\left(\log(1-q^{4n}) - \log(1-q^{4n-2})\right)\\
&= \log 2 + \frac{1}{4}\pi i z + \sum_{n\geq 1}\left(\sum_{m\geq 1}\frac{-1}{m}q^{4nm} - \sum_{m\geq 1}\frac{-1}{m}(q^{4n-2})^m)\right)\\
&= \log 2 + \frac{1}{4}\pi i z + \sum_{m\geq 1}\left(\sum_{n\geq 1}\frac{-1}{m}(q^{4m})^n + \sum_{n\geq 1}\frac{1}{m}q^{-2m}(q^{4m})^n\right)\\
&= \log 2 + \frac{1}{4}\pi i z + \sum_{m\geq 1}\frac{-1}{m\left(q^{-4m}-1\right)} + \sum_{m\geq 1}\frac{1}{m\left(q^{-2m}-q^{2m}\right)}\\
&= \log 2 + \frac{1}{4}\pi i z + \sum_{m\geq 1}\frac{-1}{m\left(e^{-4\pi i m z}-1\right)} + \sum_{m\geq 1}\frac{1}{m\left(e^{-2\pi i m z}-e^{2\pi i m z}\right)}.
\end{aligned}
\qquad (8.0.6)
$$

Also,

$$
\log\sqrt{-iz} = -\frac{\pi}{4}i + \frac{1}{2}\log z. \qquad (8.0.7)
$$

Hence, using (8.0.5)–(8.0.7), we can write (8.0.4) as

$$
\begin{aligned}
\sum_{n\geq 1}\frac{-2}{(2n-1)(e^{(2n-1)\pi i/z}-1)} &= -\frac{\pi}{4}i + \frac{1}{2}\log z + \log 2 + \frac{\pi}{4}iz\\
&+ \sum_{n\geq 1}\frac{-1}{n(e^{-4\pi i n z}-1)} + \sum_{n\geq 1}\frac{1}{n(e^{-2\pi i n z}-e^{2\pi i n z})}. \qquad (8.0.8)
\end{aligned}
$$

It suffices to prove (8.0.8) when z is purely imaginary, since the general result will then follow by analytic continuation. Let $z = it$, where t is a positive real number. Then (8.0.8) is equivalent to

$$
\sum_{n\geq 1}\frac{-2}{(2n-1)(e^{(2n-1)\pi/t)}-1} = \frac{1}{2}\log t + \log 2 - \frac{\pi}{4}t
$$

$$+\sum_{n\geq 1}\frac{-1}{n(e^{4\pi nt}-1)}+\sum_{n\geq 1}\frac{1}{n(e^{2\pi nt}-e^{-2\pi nt})}. \tag{8.0.9}$$

Recall the following relations, namely

$$\frac{1}{e^z-1}=\frac{1}{2}\coth\left(\frac{z}{2}\right)+\frac{1}{2} \tag{8.0.10}$$

and

$$\text{csch } z=\frac{1}{\sinh z}=\frac{2}{e^z-e^{-z}}. \tag{8.0.11}$$

Hence, we can write (8.0.9) as

$$\sum_{n\geq 1}\left(\frac{-2}{2n-1}\left[\frac{1}{2}\coth\left(\frac{(2n-1)\pi}{2t}\right)-\frac{1}{2}\right]+\frac{1}{n}\left[\frac{1}{2}\coth\left(2\pi nt\right)-\frac{1}{2}\right]\right.$$
$$\left.-\frac{1}{n}\left[\frac{1}{2}\text{csch}\left(2\pi nt\right)\right]\right)=\frac{1}{2}\log t+\log 2-\frac{\pi}{4}t. \tag{8.0.12}$$

Let $a=\pi/(2t)$ and $b=2\pi t$, so that $ab=\pi^2$, where a and b are positive real numbers. Then we may write (8.0.12) as

$$\sum_{n\geq 1}\left(\frac{-2}{2n-1}\left[\frac{1}{2}\coth\left((2n-1)a\right)-\frac{1}{2}\right]+\frac{1}{n}\left[\frac{1}{2}\coth\left(bn\right)-\frac{1}{2}\right]\right.$$
$$\left.-\frac{1}{n}\left[\frac{1}{2}\text{csch}\left(bn\right)\right]\right)=\frac{1}{2}\log t+\log 2-\frac{\pi}{4}t. \tag{8.0.13}$$

Utilizing the well-known equality $\text{csch}(z)=(1/2)(\coth(z/2)-\tanh(z/2))$, we write (8.0.13) in the equivalent form

$$\sum_{n\geq 1}\left(\frac{-2}{2n-1}\left[\frac{1}{2}\coth((2n-1)a)-\frac{1}{2}\right]+\frac{1}{n}\left[\frac{1}{2}\coth(bn)-\frac{1}{2}\right]\right.$$
$$\left.-\frac{1}{n}\left[\frac{1}{4}\coth\left(\frac{bn}{2}\right)-\frac{1}{4}\tanh\left(\frac{bn}{2}\right)\right]\right)=\frac{1}{2}\log t+\log 2-\frac{\pi}{4}t. \tag{8.0.14}$$

Hence,

$$\sum_{n\geq 1}\left(-\frac{1}{2n-1}\coth((2n-1)a)+\frac{1}{2n}\coth(bn)-\frac{1}{4n}\coth(\frac{bn}{2})-\frac{1}{4n}\tanh(\frac{bn}{2})\right.$$
$$\left.+\left(\frac{1}{2n-1}-\frac{1}{2n}\right)\right)=\frac{1}{2}\log t+\log 2-\frac{\pi}{4}t. \tag{8.0.15}$$

Using the equality $\sum_{n\geq 1}\left(\frac{1}{2n-1}-\frac{1}{2n}\right)=\log 2$ in (8.0.15), we deduce that (8.0.15) is equivalent to

$$\sum_{n\geq 1}\left(-\frac{1}{2n-1}\coth((2n-1)a)+\frac{1}{2n}\coth(bn)-\frac{1}{4n}\coth(\frac{bn}{2})-\frac{1}{4n}\tanh(\frac{bn}{2})\right)$$
$$=\frac{1}{2}\log t-\frac{\pi}{4}t. \tag{8.0.16}$$

Employing the partial fraction decompositions

$$\coth(\pi z)=\frac{1}{\pi z}+\frac{2z}{\pi}\sum_{m=1}^{\infty}\frac{1}{m^2+z^2} \tag{8.0.17}$$

and

$$\tanh\left(\frac{\pi z}{2}\right)=4\pi z\sum_{m=0}^{\infty}\frac{1}{(2m+1)^2\pi^2+(\pi z)^2}, \tag{8.0.18}$$

we rewrite (8.0.16) in the equivalent form

$$\sum_{n=1}^{\infty}\left(-\frac{1}{2n-1}\left[\frac{1}{(2n-1)a}+\frac{2(2n-1)a}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{m^2+\left(\frac{(2n-1)a}{\pi}\right)^2}\right]\right.$$
$$+\frac{1}{2n}\left[\frac{1}{bn}+\frac{2bn}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{m^2+\left(\frac{bn}{\pi}\right)^2}\right]-\frac{1}{4n}\left[\frac{2}{bn}+\frac{bn}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{m^2+\left(\frac{bn}{2\pi}\right)^2}\right]$$
$$\left.+\frac{1}{4n}\left[4bn\sum_{m=1}^{\infty}\frac{1}{(2m-1)^2\pi^2+(bn)^2}\right]\right)=\frac{1}{2}\log t-\frac{\pi}{4}t. \tag{8.0.19}$$

Simplifying (8.0.19), we arrive at

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(\frac{-2a}{m^2\pi^2+(2n-1)^2a^2}+\frac{b}{m^2\pi^2+n^2b^2}+\frac{-b/4}{m^2\pi^2+\left(\frac{bn}{2}\right)^2}\right.$$
$$\left.+\frac{b}{(2m-1)^2\pi^2+(bn)^2}\right)=\frac{1}{2}\log t-\frac{\pi}{4}t+\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2a}. \tag{8.0.20}$$

Using the facts that $ab=\pi^2$ and $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}=\frac{\pi^2}{8}$, we write the last line in the form

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(\frac{-2}{m^2b+(2n-1)^2a}+\frac{1}{m^2a+n^2b}-\frac{1}{(2m)^2a+n^2b}\right.$$
$$\left.+\frac{1}{(2m-1)^2a+n^2b}\right)=\frac{1}{2}\log t-\frac{\pi}{4}t+\frac{\pi^2}{8a}, \tag{8.0.21}$$

or, more simply,

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(\frac{-2}{m^2b+(2n-1)^2a}+\frac{2}{(2m-1)^2a+n^2b}\right)=\frac{1}{2}\log t-\frac{\pi}{4}t+\frac{\pi^2}{8a}. \tag{8.0.22}$$

Since $a = \pi/(2t)$ and $b = 2\pi t$, we may rewrite (8.0.22) as

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(\frac{1}{(2m-1)^2a+n^2b}-\frac{1}{m^2b+(2n-1)^2a}\right)=\frac{1}{8}\log\frac{b}{a}-\frac{1}{4}\log 2. \tag{8.0.23}$$

Next, set $a = \pi e^{-\gamma}$ and $b = \pi e^{\gamma}$. Then (8.0.23) is equivalent to

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(\frac{1}{(2m-1)^2e^{-\gamma}+n^2e^{\gamma}}-\frac{1}{m^2e^{\gamma}+(2n-1)^2e^{-\gamma}}\right)=\frac{\pi}{4}\gamma-\frac{\pi}{4}\log 2. \tag{8.0.24}$$

We now prove (8.0.24). Let a_{mn} denote the summands in (8.0.24). Observe that $a_{mn} = -a_{nm}$. Hence,

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}a_{mn}=\lim_{n\to\infty}\sum_{\nu=1}^{n}\left(\sum_{\mu=1}^{n}+\sum_{\mu=n+1}^{\infty}a_{\mu\nu}\right)=\lim_{n\to\infty}\sum_{\nu=1}^{n}\sum_{\mu=n+1}^{\infty}a_{\mu\nu}. \tag{8.0.25}$$

For $\mu \geq 2$, the inequalities

$$\frac{1}{(2\mu-1)^2e^{-\gamma}+\nu^2e^{\gamma}}<\int_{\mu-1}^{\mu}\frac{dx}{(2\mu-1)^2e^{-\gamma}+\nu^2e^{\gamma}}<\frac{1}{(2\mu-3)^2e^{-\gamma}+\nu^2e^{\gamma}}$$

give, on summing over μ, $n+1 \leq \mu < \infty$,

$$\begin{aligned}0&<\int_{n}^{\infty}\frac{dx}{(2x-1)^2e^{-\gamma}+\nu^2e^{\gamma}}-\sum_{\mu=n+1}^{\infty}\frac{1}{(2\mu-1)^2e^{-\gamma}+\nu^2e^{\gamma}}\\&<\frac{1}{(2n-1)^2e^{-\gamma}+\nu^2e^{\gamma}}\\&<\frac{e^{\gamma}}{(2n-1)^2}.\end{aligned} \tag{8.0.26}$$

Evaluating the integral and summing over ν, $1 \leq \nu \leq n$, we see that

$$0<\frac{1}{2}\sum_{\nu=1}^{n}\frac{1}{\nu}\arctan\left(\frac{\nu e^{\gamma}}{2n-1}\right)-\sum_{\nu=1}^{n}\sum_{\mu=n+1}^{\infty}\frac{1}{(2\mu-1)^2e^{-\gamma}+\nu^2e^{\gamma}}<\frac{e^{\gamma}}{(2n-1)^2}n. \tag{8.0.27}$$

Interpreting

$$\frac{1}{2}\sum_{\nu=1}^{n}\frac{1}{\nu}\arctan\left(\frac{\nu e^{\gamma}}{2n-1}\right)=\frac{1}{2}\sum_{\nu=1}^{n}\frac{2n-1}{\nu e^{\gamma}}\arctan\left(\frac{\nu e^{\gamma}}{2n-1}\right)\frac{e^{\gamma}}{2n-1}$$

as a Riemann sum for

$$\frac{1}{2}\int_{0}^{\frac{ne^{\gamma}}{2n-1}}\frac{\arctan(x)}{x}\,dx,$$

and letting n tend to ∞ in (8.0.27), we conclude that

$$\lim_{n\to\infty}\sum_{\nu=1}^{n}\sum_{\mu=n+1}^{\infty}\frac{1}{(2\mu-1)^2e^{-\gamma}+\nu^2e^{\gamma}}=\frac{1}{2}\int_{0}^{\frac{e^{\gamma}}{2}}\frac{\arctan(x)}{x}\,dx. \tag{8.0.28}$$

Similarly, the inequalities

$$\frac{1}{\mu^2 e^\gamma + (2\nu-1)^2 e^{-\gamma}} < \int_{\mu-1}^{\mu} \frac{dx}{x^2 e^\gamma + (2\nu-1)^2 e^{-\gamma}} < \frac{1}{(\mu-1)^2 e^\gamma + (2\nu-1)^2 e^{-\gamma}}$$

give, on summing over μ, $n+1 \le \mu < \infty$,

$$\begin{aligned} 0 &< \int_n^\infty \frac{dx}{x^2 e^\gamma + (2\nu-1)^2 e^{-\gamma}} - \sum_{\mu=n+1}^{\infty} \frac{1}{\mu^2 e^\gamma + (2\nu-1)^2 e^{-\gamma}} \\ &< \frac{1}{n^2 e^\gamma + (2\nu-1)^2 e^{-\gamma}} \\ &< \frac{e^{-\gamma}}{n^2}. \end{aligned}$$

Evaluating the integral and summing over ν, $1 \le \nu \le n$, we deduce that

$$0 < \sum_{\nu=1}^{n} \frac{1}{2\nu-1} \arctan\left(\frac{2\nu-1}{ne^\gamma}\right) - \sum_{\nu=1}^{n} \sum_{\mu=n+1}^{\infty} \frac{1}{\mu^2 e^\gamma + (2\nu-1)^2 e^{-\gamma}} < \frac{e^{-\gamma}}{n}. \tag{8.0.29}$$

Interpreting

$$\sum_{\nu=1}^{n} \frac{1}{2\nu-1} \arctan\left(\frac{2\nu-1}{ne^\gamma}\right) = \frac{1}{2} \sum_{\nu=1}^{n} \frac{ne^\gamma}{2\nu-1} \arctan\left(\frac{2\nu-1}{ne^\gamma}\right) \frac{2}{ne^\gamma}$$

as a Riemann sum for

$$\frac{1}{2} \int_0^{\frac{2n-1}{ne^\gamma}} \frac{\arctan(x)}{x}\, dx,$$

we see that

$$\lim_{n\to\infty} \sum_{\nu=1}^{n} \sum_{\mu=n+1}^{\infty} \frac{1}{\mu^2 e^\gamma + (2\nu-1)^2 e^{-\gamma}} = \frac{1}{2} \int_0^{2e^{-\gamma}} \frac{\arctan(x)}{x}\, dx. \tag{8.0.30}$$

Subtracting (8.0.30) from (8.0.28) and utilizing (8.0.25), we find that

$$\begin{aligned} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{mn} &= \lim_{n\to\infty} \sum_{\nu=1}^{n} \sum_{\mu=n+1}^{\infty} \left(\frac{1}{(2\mu-1)^2 e^{-\gamma} + \nu^2 e^\gamma} - \frac{1}{\mu^2 e^\gamma + (2\nu-1)^2 e^{-\gamma}} \right) \\ &= \frac{1}{2} \int_0^{\frac{e^\gamma}{2}} \frac{\arctan(x)}{x}\, dx - \frac{1}{2} \int_0^{2e^{-\gamma}} \frac{\arctan(x)}{x}\, dx \\ &= \frac{1}{2} \int_{2e^{-\gamma}}^{\frac{e^\gamma}{2}} \frac{\arctan(x)}{x}\, dx \\ &= \frac{1}{2} \int_{-(\gamma-\log 2)}^{\gamma-\log 2} \arctan(e^t)\, dt \\ &= \frac{1}{4} \int_{-(\gamma-\log 2)}^{\gamma-\log 2} \arctan(e^t) + \arctan(e^{-t})\, dt \\ &= \frac{1}{4} \int_{-(\gamma-\log 2)}^{\gamma-\log 2} \frac{\pi}{2}\, dt \end{aligned}$$

$$= \frac{\pi}{4}\gamma - \frac{\pi}{4}\log 2, \tag{8.0.31}$$

which completes the proof of (8.0.24). $\square$

Remark. Applying this method as above to the transformation

$$\Theta_2(-\frac{1}{z}) = \sqrt{-iz}\Theta_4(z) \tag{8.0.32}$$

leads to (8.0.24) again, so that (8.0.32) easily follows by the work above.

References

[1] G.E. Andrews, *A generalization of the Göllnitz–Gordon partition theorems*, Proc. Amer. Math. Soc. **18** (1967), 945–952.

[2] G.E. Andrews, *On q-difference equations for certain well-poised basic hypergeometric series*, Quart. J. Math. (Oxford) **19** (1968), 433–447.

[3] G.E. Andrews, *On the General Rogers–Ramanujan Theorem*, Memoir Amer. Math. Soc. **152** (1974), No. 152.

[4] G.E. Andrews, *The Theory of Partitions*. Addison–Wesley, Reading, MA, 1976.

[5] G.E. Andrews, *Ramanujan's "lost" notebook III. The Rogers–Ramanujan continued fraction*, Adv. Math. **41** (1981), 186–208.

[6] G.E. Andrews, *Rogers–Ramanujan identities for two-color partitions*, Indian J. of Math. **29** (1978), 117–125.

[7] G.E. Andrews and B.C. Berndt, *Ramanujan's Lost Notebook, Part I.* Springer, New York, 2005.

[8] G.E. Andrews, B.C. Berndt, L. Jacobsen, and R.L. Lamphere, *The Continued Fractions Found in the Unorganized Portions of Ramanujan's Notebooks*, Memoir Amer. Math. Soc. **99** (1992), No. 477.

[9] G.E. Andrews and P. Paule, *MacMahon's partition analysis XI: Hexagonal plane partitions*, Acta Arith. **126.3** (2007), 281–294.

[10] N.D. Baruah and J. Bora, *New proofs of Ramanujan's modular equations of degree 9*, Indian J. of Mathematics, Vol. **47** (2005), No. 1, 99–122.

[11] N.D. Baruah and J. Bora, *Further analogues of the Rogers–Ramanujan functions with applications to partitions*, Combinatorial Number Theory, de Gruyter, Berlin (2007), 81–101.

[12] N.D. Baruah and J. Bora, *Modular relations for the nonic analogues of the Rogers–Ramanujan functions with applications to partitions*, J. Number Theory **128** No. 1, (2008), 175-206.

[13] N.D. Baruah, J. Bora, and N. Saikia, *Some new proofs of modular relations for the Göllnitz–Gordon functions*, Ramanujan J. **15** (2008), 281–301.

[14] B.C. Berndt, *Ramanujan's Notebooks III.* Springer-Verlag, New York, 1991.

[15] B.C. Berndt, *Ramanujan's Notebooks IV.* Springer-Verlag, New York, 1994.

[16] B.C. Berndt, *Ramanujan's Notebooks V*. Springer-Verlag, New York, 1998.

[17] B.C. Berndt, H.H. Chan, and S.–S. Huang, *Incomplete elliptic integrals in Ramanujan's lost notebook*, in *q-Series from a Contemporary Perspective*, M.E.H. Ismail and D. Stanton, eds., American Mathematical Society, Providence, RI, 2000, pp. 79–126.

[18] B.C. Berndt, H.H. Chan, S.–S. Huang, S.–Y. Kang, J. Sohn, and S.H. Son, *The Rogers–Ramanujan continued fraction*, J. Comp. Appl. Math. **105** (1999), 9–24.

[19] B.C. Berndt, G. Choi, Y.–S. Choi, H. Hahn, B.P. Yeap, A.J. Yee, H. Yesilyurt, and J. Yi, *Ramanujan's Forty Identities for the Rogers–Ramanujan Functions*, Memoir Amer. Math. Soc. **188** (2007), No. 880.

[20] B.C. Berndt, C. Gugg, S. Kongsiriwong, and J. Thiel, *A simple proof of the theta inversion formulas* in *Ramanujan Rediscovered: Proceedings of a Conference on Elliptic Functions, Partitions, and q-Series in Memory of K. Venkatachaliengar: Bangalore, 1-5 June*, to appear.

[21] B.C. Berndt, S.–S. Huang, J. Sohn, and S.H. Son, *Some theorems on the Rogers–Ramanujan continued fraction in Ramanujan's lost notebook*, Trans. Amer. Math. Soc. **352** (2000), 2157–2177.

[22] B.C. Berndt and K. Venkatachaliengar, *On the transformation formula for the Dedekind-eta function*, in *Symbolic Computation, Number Theory, Special Functions, Physics, and Combinatorics*, F.G. Garvan and M.E.H. Ismail, eds., Kluwer, Dordrecht, 2001, 73–77.

[23] B. C. Berndt and H. Yesilyurt, *New identities for the Rogers–Ramanujan functions*, Acta Arith. **120** (2005), 395–413.

[24] A.J.F. Biagioli, *A proof of some identities of Ramanujan using modular forms*, Glasgow Math. J. **31** (1989), 271–295.

[25] B.J. Birch, *A look back at Ramanujan's notebooks*, Math. Proc. Cambridge Philos. Soc. **78** (1975), 73–79.

[26] R. Blecksmith, J. Brillhart, and I. Gerst, *A fundamental modular identity and some applications*, Math. Comp. **61** (1993), 83–95.

[27] D. Bressoud, *Proof and Generalization of Certain Identities Conjectured by Ramanujan*, Ph.D. Thesis, Temple University, 1977.

[28] D. Bressoud, *Some identities involving Rogers–Ramanujan-type functions*, J. London Math. Soc. (2) **16** (1977), 9–18.

[29] K. Bringmann and H. Swisher, *On a conjecture of Koike on identities between Thompson series and Rogers–Ramanujan functions*, Proc. Amer. Math. Soc. **135** (2007), no. 8, 2317–2326.

[30] K. Bringmann and H. Swisher, *Erratum to "On a conjecture of Koike on identities between Thompson series and Rogers–Ramanujan functions"*, Proc. Amer. Math. Soc. **136** (2008), no. 4, 1501.

[31] Z. Cao, *Product Identities for Theta Functions*, Ph.D. thesis, University of Illinois at Urbana-Champaign, 2008.

[32] Z. Cao, *Integer matrix exact covering systems and product identities for theta functions*, Preprint.

[33] H.–C. Chan and S. Ebbing, *Factorization theorems for the Rogers–Ramanujan continued fraction in the Lost Notebook*, preprint.

[34] H.H. Chan and S.–S. Huang, *On the Ramanujan–Göllnitz–Gordon continued fraction*, Ramanujan J. **1** (1997), 75–90.

[35] S.–L. Chen and S.–S. Huang, *New modular relations for the Göllnitz–Gordon functions*, J. Number Theory, **93** (2002), 58–75.

[36] B. Cho, J. K. Koo, Y. K. Park, *Arithmetic of the Ramanujan–Göllnitz–Gordon continued fraction*, J. Number Thy. **129** (2009), 922-947.

[37] W. Chu, *Common source of numerous theta function identities*, Glasgow Math. J. **49** (2007), 61–79.

[38] S. Cooper, *The quintuple product identity*, Int. J. Number Theory, **2** (2006), no. 1, 115–161.

[39] H.B.C. Darling, *Proofs of certain identities and congruences enunciated by S. Ramanujan*, Proc. London Math. Soc. (2) **19** (1921), 350–372.

[40] E. Fiedler, *Ueber eine besondere Classe irrationaler Modulargleichungen der elliptischen Functionen*, Vierteljahrsschr. Naturforsch. Gesell. (Zürich) **12** (1885), 129–229.

[41] N.J. Fine, *Basic Hypergeometric Series and Applications*, Mathematical Surveys and Monographs 27, Amer. Math. Soc., 1988.

[42] H. Göllnitz, *Einfache Partitionen*, Diplomarbeit W.S., Göttingen, 1960 (unpublished).

[43] H. Göllnitz, *Partitionen mit Differenzenbedingungen*, J. Reine Angew. Math. **225** (1967), 154–190.

[44] B. Gordon, *Some Ramanujan-like continued fractions*, Abstracts of Short Communications, International Congress of Mathematicians, Stockholm, 1962, pp. 29–30.

[45] B. Gordon, *Some continued fractions of the Rogers–Ramanujan type*, Duke Math. J. **32** (1965), 741–748.

[46] B. Gordon and R.J. McIntosh, *Modular transformations of Ramanujan's fifth and seventh order mock theta functions*, Ramanujan J. **7** (2003), 193–222.

[47] C. Guetzlaff, *Aequatio modularis pro transformatione functionum ellipticarum septimi ordinis*, J. Reine Angew. Math. **12** (1834), 173–177.

[48] C. Gugg, *Two modular equations for squares of the Rogers–Ramanujan functions with applications*, Ramanujan J. **18** (2009), 183–207.

[49] C. Gugg, *A new proof of Ramanujan's modular equation relating $R(q)$ with $R(q^5)$*, Ramanujan J. **20** (2009), 163–177.

[50] C. Gugg, *Modular identities involving cubes of the Rogers–Ramanujan functions, analogues, and applications*, Submitted for publication.

[51] H. Hahn, *Septic analogues of the Rogers–Ramanujan functions*, Acta Arith. **110** (2003), 381–391.

[52] D. Hickerson, *A proof of the mock theta conjectures*, Invent. Math. **94** (1988), 639–660.

[53] M.D. Hirschhorn, *On the expansion of Ramanujan's continued fraction*, Ramanujan J. **2** (1998), no. 4, 521–527.

[54] M.D. Hirschhorn, *An identity of Ramanujan, and applications*, Contemporary Math. **254** (2000), 229–234.

[55] M.D. Hirschhorn and D.C. Hunt, *A simple proof of the Ramanujan conjecture for powers of 5*, J. Reine Angew. Math. **326** (1981), 1–17.

[56] M.D. Hirschhorn and J.A. Sellers, *On recent congruence results of Andrews and Paule for broken k-diamond partitions*, Bulletin of the Australian Math. Soc. **75** (2007), 121–126.

[57] S.–S. Huang, *On the Rogers–Ramanujan and Ramanujan–Göllnitz–Gordon Continued Fractions*, Ph.D. thesis, University of Illinois at Urbana-Champaign, 1997.

[58] S.–S. Huang, *On modular relations for the Göllnitz–Gordon functions with applications to partitions*, J. Number Thy. **68** (1998), 178–216.

[59] S.–Y. Kang, *Some theorems on the Rogers–Ramanujan continued fraction and associated theta functions in Ramanujan's lost notebook*, Ramanujan J. **3** (1999), 91–111.

[60] S.–Y. Kang, *Ramanujan's formulas for the explicit evaluation of the Rogers–Ramanujan continued fraction and theta–functions*, Acta Arith. **90** (1999), 49–68.

[61] F. Klein, *Ueber die Transformation der elliptischen Functionen und die Auflösung der Gleichungen fünften Grades*, Math. Ann. **14** (1879), 111-172.

[62] S. Kim, *A bijective proof of the quintuple product identity*, Int. J. Number Theory, to appear.

[63] M. Koike, *Thompson series and Ramanujan's identities*, in *Galois Theory and Modular Forms*, K. Hashimoto, K. Miyake, and H. Nakamura, eds., Kluwer, Dordrecht, 2003, pp. 367–374.

[64] S. Kongsiriwong, *Infinite Series Identities in the Theory of Elliptic Functions and q-Series*, Ph.D. Thesis, University of Illinois at Urbana-Champaign, 2003.

[65] R. Lewis and Z.–G. Liu, *On two identities of Ramanujan*, Ramanujan J. **3** (1999), 335–338.

[66] M.S. Mahadeva Naika, *Some theorems on Ramanujan's cubic continued fraction and related identities*, Tamsu Oxford J. of Math. Sciences, **24** (3) (2008), 243–256.

[67] L.J. Mordell, *Note on certain modular relations considered by Messrs. Ramanujan, Darling, and Rogers*, Proc. London Math. Soc. (2) **20** (1922), 408–416.

[68] K.G. Ramanathan, *On Ramanujan's continued fraction*, Acta Arith. **43** (1984), 209–226.

[69] K.G. Ramanathan, *On the Rogers–Ramanujan continued fraction*, Proc. Indian Acad. Sci. (Math. Sci.) **93** (1984), 67–77.

[70] K.G. Ramanathan, *Ramanujan's continued fraction*, Indian J. Pure Appl. Math. **16** (1985), 695–724.

[71] K.G. Ramanathan, *Some applications of Kronecker's limit formula*, J. Indian Math. Soc. **52** (1987), 71–89.

[72] S. Ramanujan, *Proof of certain identities in combinatory analysis*, Proc. Cambridge Philos. Soc. **19** (1919), 214–216.

[73] S. Ramanujan, *Some properties of $p(n)$, the number of partitions of n*, Proc. of the Camb. Phil. Soc., **XIX** (1919), 207–210.

[74] S. Ramanujan, *Algebraic relations between certain infinite products*, Proc. London Math. Soc. **2** (1920), p. xviii.

[75] S. Ramanujan, *Collected Papers*, Cambridge University Press, Cambridge, 1927; reprinted by Chelsea, New York, 1962; reprinted by the American Mathematical Society, Providence, RI, 2000.

[76] S. Ramanujan, *Notebooks, (2 volumes)*, Tata Instititute of Fundamental Research, Bombay, 1957.

[77] S. Ramanujan, *Collected Papers of Srinivasa Ramanujan*, Chelsea Publishing Co., New York, 1962. Edited with notes by G.H. Hardy, P.V. Aiyar, and B.M. Wilson; reprinted by the American Mathematical Society, Providence, RI, 2000.

[78] S. Ramanujan, *The Lost Notebook and Other Unpublished Papers*, Narosa, New Delhi, 1988.

[79] R.A. Rankin, *Modular Forms and Functions*, Cambridge University Press, Cambridge, 1977.

[80] S. Robins, *Arithmetic Properties of Modular Forms*, Ph.D. thesis, University of California at Los Angeles, 1991.

[81] L.J. Rogers, *Second memoir on the expansion of certain infinite products*, Proc. London Math. Soc. **25** (1894), 318–343.

[82] L.J. Rogers, *On a type of modular relation*, Proc. London Math. Soc. **19** (1921), 387–397.

[83] R. Russell, *On $\kappa\lambda - \kappa'\lambda'$ modular equations*, Proc. London Math. Soc. **(2)** 19 (1920), 387–397.

[84] L. Schläfli, *Bewies der Hermiteschen Verwandlungsstafeln für die elliptischen Modularfunctionen*, J. Reine Angew. Math. **72** (1870), 360–369.

[85] B. Schoeneberg, *Elliptic Modular Functions*, Springer-Verlag, New York, 1974.

[86] H. Schröter, *De aequationibus modularibus*, Dissertatio Inauguralis, Albertina Litterarum Universitate, 1854 Regiomonti.

[87] H. Schröter, *Extrait d'une lettre adressee A.M. Liouville*, J. Math. **(2)** 3 (1858), 258–264.

[88] H. Schröter, *Beiträge zur Theorie der Elliptischen Funktionen*, Acta Math. **5** (1894), 205–208.

[89] L.J. Slater, *Further identities of the Rogers–Ramanujan type*, Proc. London Math. Soc. (2) **54** (1952), 147–167.

[90] L.A. Sohncke, *Aequationes modulares pro transformatione functionum ellipticarum et undecimi et decimi tertii et decimi septimi ordinis*, J. Reine Angew. Math. **12** (1834), 178.

[91] J. Sohn, q-Continued Fractions and Related q-Series, Ph.D. thesis, University of Illinois at Urbana-Champaign, 2001.

[92] L.A. Sohncke, *Aequationes modulares pro transformatione functionum ellipticarum*, J. Reine Angew. Math. **16** (1837), 97–130.

[93] M. Somos, Email September 15, 2008.

[94] M. Somos, private communications, October, 2008.

[95] S.H. Son, *Some theta function identities related to the Rogers–Ramanujan continued fraction*, Proc. Amer. Math. Soc., **126** (1998), 2895–2902.

[96] K.R. Vasuki and B.R. Srivatsa Kumar, *Certain identities for the Ramanujan–Göllnitz–Gordon continued fraction*, J. of Comp. and Applied Mathematics, **187** (2006), 87–95.

[97] G.N. Watson, *Theorems stated by Ramanujan (VII): Theorems on continued fractions*, J. London Math. Soc. **4** (1929), 39–48.

[98] G.N. Watson, *Theorems stated by Ramanujan (IX): Two continued fractions*, J. London Math. Soc. **4** (1929) 231–237.

[99] G.N. Watson, *Proof of certain identities in combinatory analysis*, J. Indian Math. Soc., **20** (1933), 57–69.

[100] E.T. Whittaker and G.N. Watson, *A Course of Modern Analysis*, 4^{th} ed., University Press, Cambridge, 1966.

[101] H. Yesilyurt, *A generalization of a modular identity of Rogers*, J. Number Thy. **129** (2009), 1256–1271.

[102] J. Yi, *Modular equations for the Rogers–Ramanujan continued fraction and the Dedekind eta-function*, Ramanujan J. **5** (2001), No. 4, 377–384.